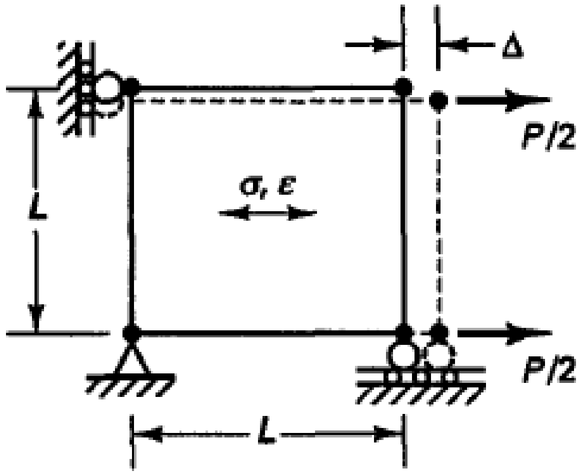
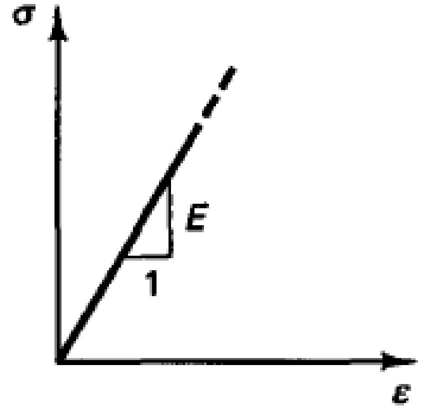
Geometric Non-Linearity and Total Lagrangian

Classification of analyses

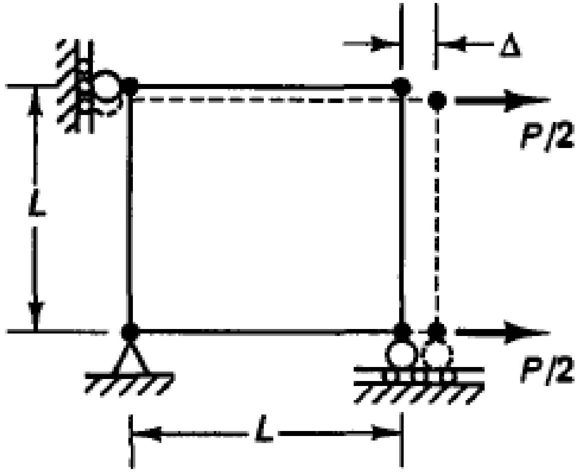


$\sigma = P/A$
$\varepsilon = \sigma/E$
$\Delta = \varepsilon L$

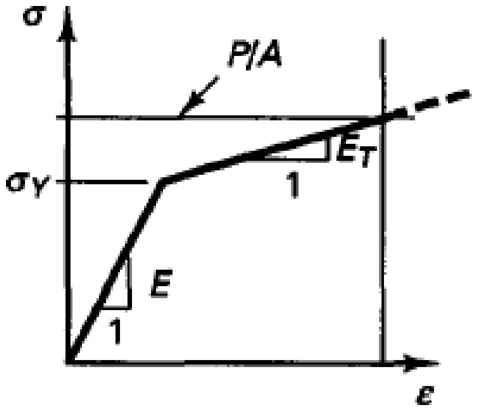


$\varepsilon < 0.04$

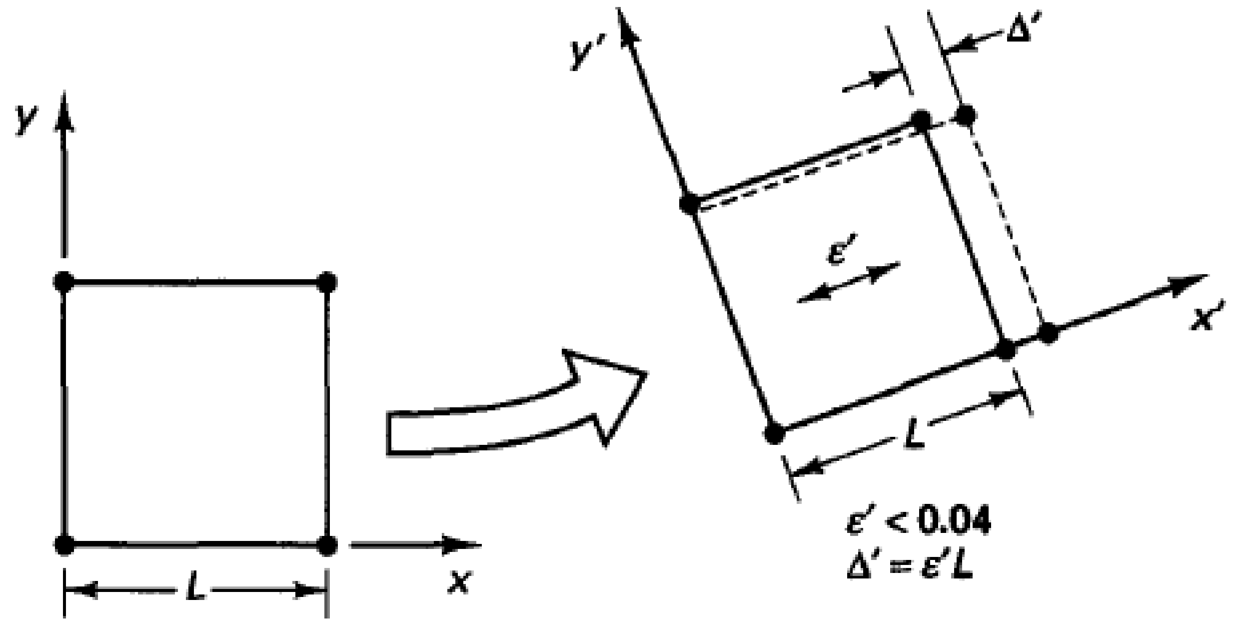
(a) Linear elastic (infinitesimal displacements)



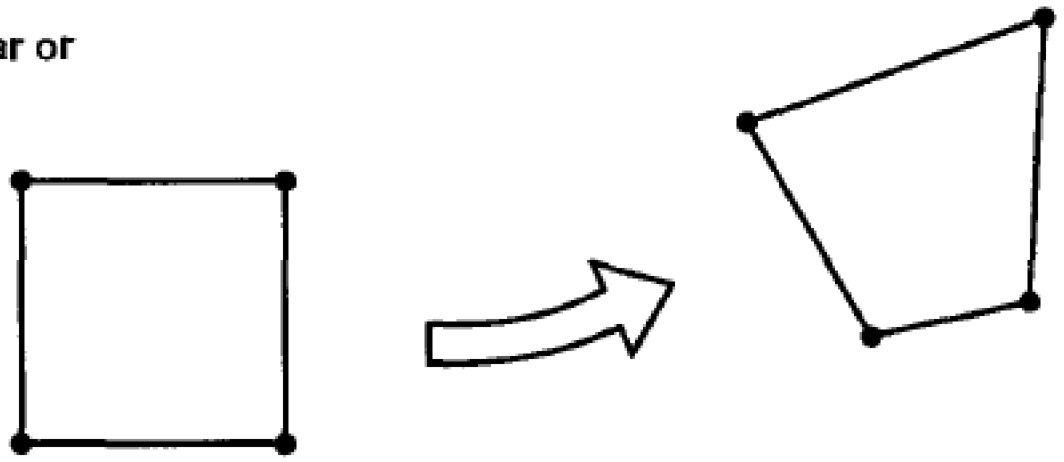
$\sigma = P/A$
$\varepsilon = \dfrac{\sigma_Y}{E} + \dfrac{\sigma - \sigma_Y}{E_T}$
$\varepsilon < 0.04$



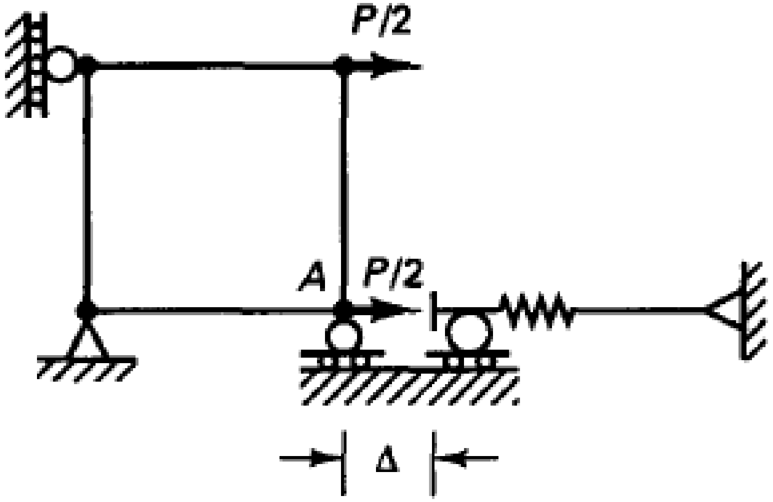
(b) Materially-nonlinear-only (infinitesimal displacements, but nonlinear stress-strain relation)



(c) Large displacements and large rotations but small strains. Linear or nonlinear material behavior



(d) Large displacements, large rotations, and large strains. Linear or nonlinear material bahavior



(e) Change in boundary condition at displacement Δ

EXAMPLE 6.1: A bar rigidly supported at both ends is subjected to an axial load as shown in Fig. E6.1(a). The stress-strain relation and the load-versus-time curve relation are given in Figs. E6.1(b) and (c), respectively. Assuming that the displacements and strains are small and that the load is applied slowly, calculate the displacement at the point of load application.

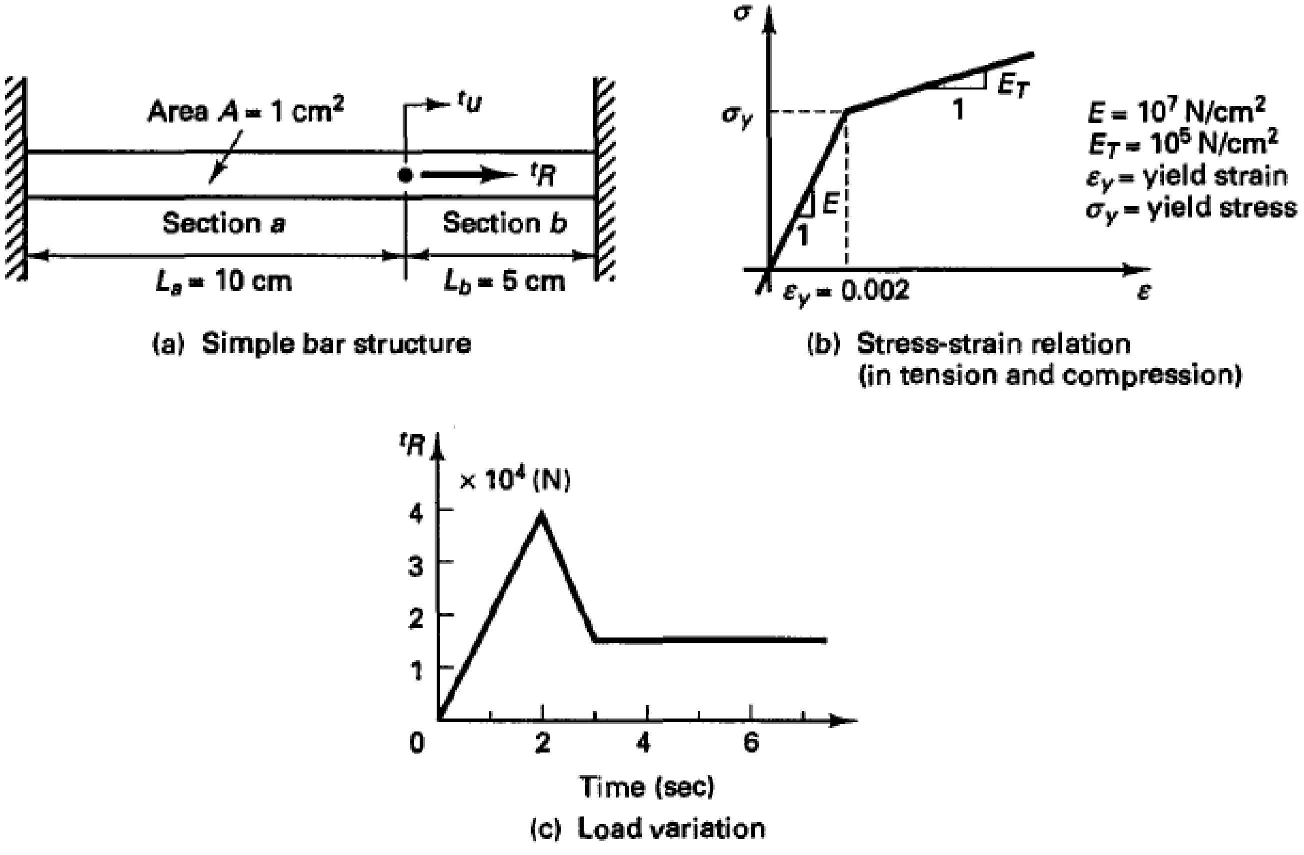


Figure E6.1 Analysis of simple bar structure

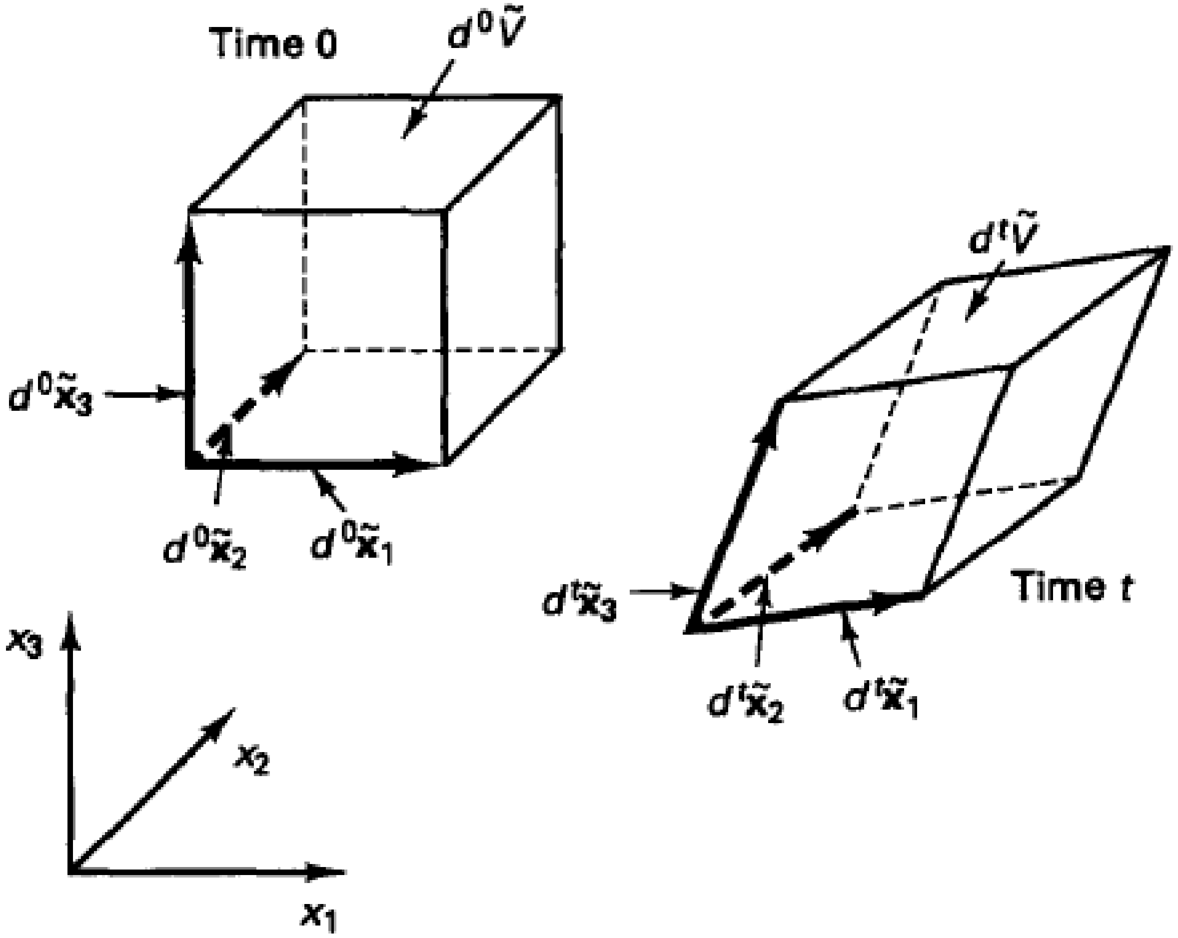


Figure E6.5 Infinitesimal volumes at times 0 and t

Deformation Gradient

$$\boxed{{}_0^t\mathbf{X} = \begin{bmatrix} \dfrac{\partial\,{}^tx_1}{\partial\,{}^0x_1} & \dfrac{\partial\,{}^tx_1}{\partial\,{}^0x_2} & \dfrac{\partial\,{}^tx_1}{\partial\,{}^0x_3} \\ \dfrac{\partial\,{}^tx_2}{\partial\,{}^0x_1} & \dfrac{\partial\,{}^tx_2}{\partial\,{}^0x_2} & \dfrac{\partial\,{}^tx_2}{\partial\,{}^0x_3} \\ \dfrac{\partial\,{}^tx_3}{\partial\,{}^0x_1} & \dfrac{\partial\,{}^tx_3}{\partial\,{}^0x_2} & \dfrac{\partial\,{}^tx_3}{\partial\,{}^0x_3} \end{bmatrix}}$$

$$ {}_0^t\mathbf{X} = ({}_0\boldsymbol{\nabla}\,{}^t\mathbf{x}^T)^T$$

where ${}_0\boldsymbol{\nabla}$ is the gradient operator

$${}_0\boldsymbol{\nabla} = \begin{bmatrix} \dfrac{\partial}{\partial\,{}^0x_1} \\ \dfrac{\partial}{\partial\,{}^0x_2} \\ \dfrac{\partial}{\partial\,{}^0x_3} \end{bmatrix}; \qquad {}^t\mathbf{x}^T = [{}^tx_1 \quad {}^tx_2 \quad {}^tx_3]$$

__Example__

The ~~displacement~~ deformed coord. of a ~~vector~~ point are related to its original coordinates by the relation

$$ {}^{t}x_1 = {}^{0}X_1 \quad {}^{t}x_2 = {}^{0}X_2 + 2\,{}^{0}X_3 \quad {}^{t}x_3 = {}^{0}X_3 + 2\,{}^{0}X_2 $$

determine the deformation gradient

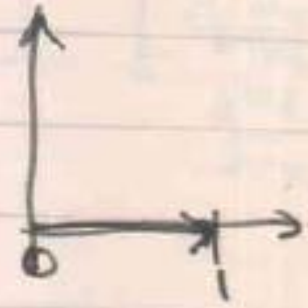
Determine the deformed shape of fiber represente[d] by vector $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$

$$\underset{\sim}{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

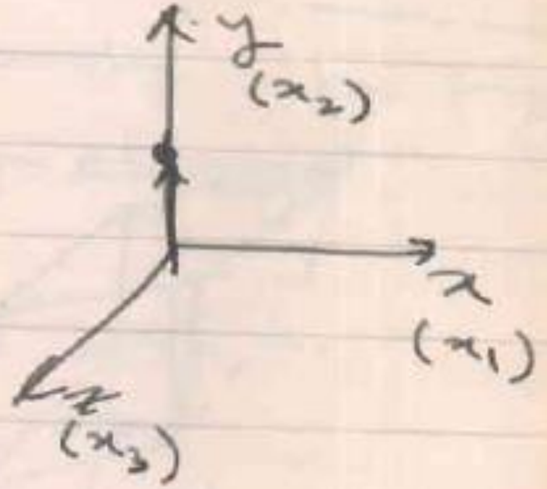
$$d^t \underset{\sim}{x} = \underset{\sim}{F} \cdot d^0 \underset{\sim}{x} \qquad \left(d^t x_1 \underset{\sim}{e_1} + d^t x_2 \underset{\sim}{e_2} + d^t x_3 \underset{\sim}{e_3} \right) =$$

$$\begin{Bmatrix} d^t x_1 \\ d^t x_2 \\ d^t x_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{Bmatrix} d^0 x_1 \\ d^0 x_2 \\ d^0 x_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

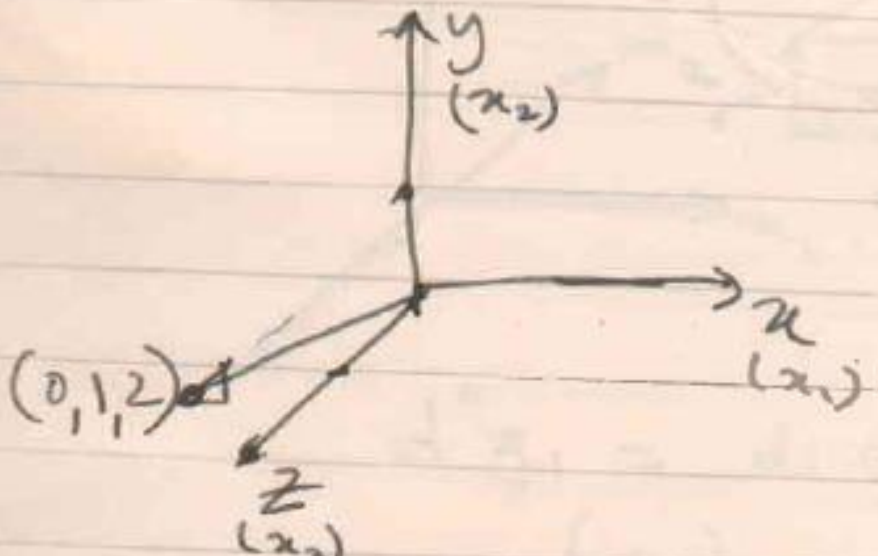
$$d^t \vec{x} = 1\,\vec{e}_1 + 0\,\vec{e}_2 + 0\,\vec{e}_3$$



$$\begin{Bmatrix} d^t x_1 \\ d^t x_2 \\ d^t x_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}$$



$$d^t \vec{x} = 0\,\vec{e}_1 + \vec{e}_2 + 2\,\vec{e}_3$$



EXAMPLE 6.6: Consider the element in Fig. E6.6. Evaluate the deformation gradient and the mass density corresponding to the configuration at time t.

The displacement interpolation functions for this element were given in Fig. 5.4. Since the ${}^0x_1$, ${}^0x_2$ axes correspond to the r, s axes, respectively, we have

$$h_1 = \tfrac{1}{4}(1 + {}^0x_1)(1 + {}^0x_2); \qquad h_2 = \tfrac{1}{4}(1 - {}^0x_1)(1 + {}^0x_2)$$

$$h_3 = \tfrac{1}{4}(1 - {}^0x_1)(1 - {}^0x_2); \qquad h_4 = \tfrac{1}{4}(1 + {}^0x_1)(1 - {}^0x_2)$$

and

$$\frac{\partial h_1}{\partial {}^0x_1} = \frac{1}{4}(1 + {}^0x_2); \qquad \frac{\partial h_2}{\partial {}^0x_1} = -\frac{1}{4}(1 + {}^0x_2)$$

$$\frac{\partial h_3}{\partial {}^0x_1} = -\frac{1}{4}(1 - {}^0x_2); \qquad \frac{\partial h_4}{\partial {}^0x_1} = \frac{1}{4}(1 - {}^0x_2)$$

$$\frac{\partial h_1}{\partial {}^0x_2} = \frac{1}{4}(1 + {}^0x_1); \qquad \frac{\partial h_2}{\partial {}^0x_2} = \frac{1}{4}(1 - {}^0x_1)$$

$$\frac{\partial h_3}{\partial {}^0x_2} = -\frac{1}{4}(1 - {}^0x_1); \qquad \frac{\partial h_4}{\partial {}^0x_2} = -\frac{1}{4}(1 + {}^0x_1)$$

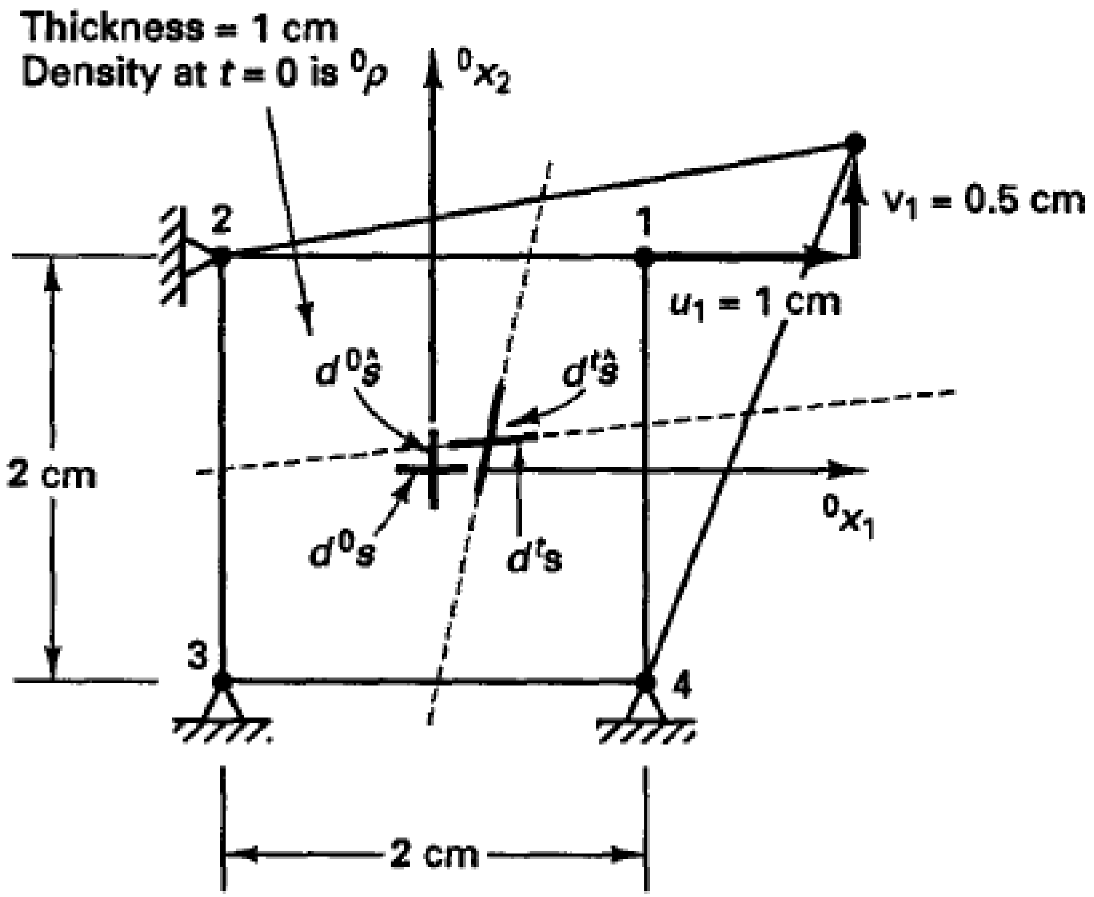


Figure E6.6 Four-node element subjected to large deformations

Now we use

$$ {}^t x_i = \sum_{k=1}^{4} h_k \, {}^t x_i^k $$

and hence,

$$ \frac{\partial\, {}^t x_i}{\partial\, {}^0 x_j} = \sum_{k=1}^{4} \left(\frac{\partial h_k}{\partial\, {}^0 x_j} \right) {}^t x_i^k $$

The nodal point coordinates at time t are

$$ {}^t x_1^1 = 2; \qquad {}^t x_2^1 = 1.5; \qquad {}^t x_1^2 = -1; \qquad {}^t x_2^2 = 1 $$

$$ {}^t x_1^3 = -1; \qquad {}^t x_2^3 = -1; \qquad {}^t x_1^4 = 1; \qquad {}^t x_2^4 = -1 $$

Hence,

$$\frac{\partial' x_1}{\partial^0 x_1} = \frac{1}{4}[(1 + {}^0x_2)(2) - (1 + {}^0x_2)(-1) - (1 - {}^0x_2)(-1) + (1 - {}^0x_2)(1)]$$

$$= \frac{1}{4}(5 + {}^0x_2)$$

and

$$\frac{\partial' x_1}{\partial^0 x_2} = \frac{1}{4}(1 + {}^0x_1); \qquad \frac{\partial' x_2}{\partial^0 x_1} = \frac{1}{8}(1 + {}^0x_2)$$

$$\frac{\partial' x_2}{\partial^0 x_2} = \frac{1}{8}(9 + {}^0x_1)$$

so that the deformation gradient is

$$
{}_0^t\mathbf{X} = \frac{1}{4}\begin{bmatrix} (5 + {}^0x_2) & (1 + {}^0x_1) \\ \frac{1}{2}(1 + {}^0x_2) & \frac{1}{2}(9 + {}^0x_1) \end{bmatrix}
$$

and using (6.26), the mass density in the deformed configuration is

$$
{}^t\rho = \frac{32\,{}^0\rho}{(5 + {}^0x_2)(9 + {}^0x_1) - (1 + {}^0x_1)(1 + {}^0x_2)}
$$

EXAMPLE 6.7: The *stretch* ${}^t\lambda$ of a line element of a general body in motion is defined as ${}^t\lambda = d^ts/d^0s$, where d^0s and d^ts are the original and current lengths of the line element as shown in Fig. E6.7. Prove that ${}^t\lambda = ({}^0\mathbf{n}^T\,{}^t_0\mathbf{C}\,{}^0\mathbf{n})^{1/2}$

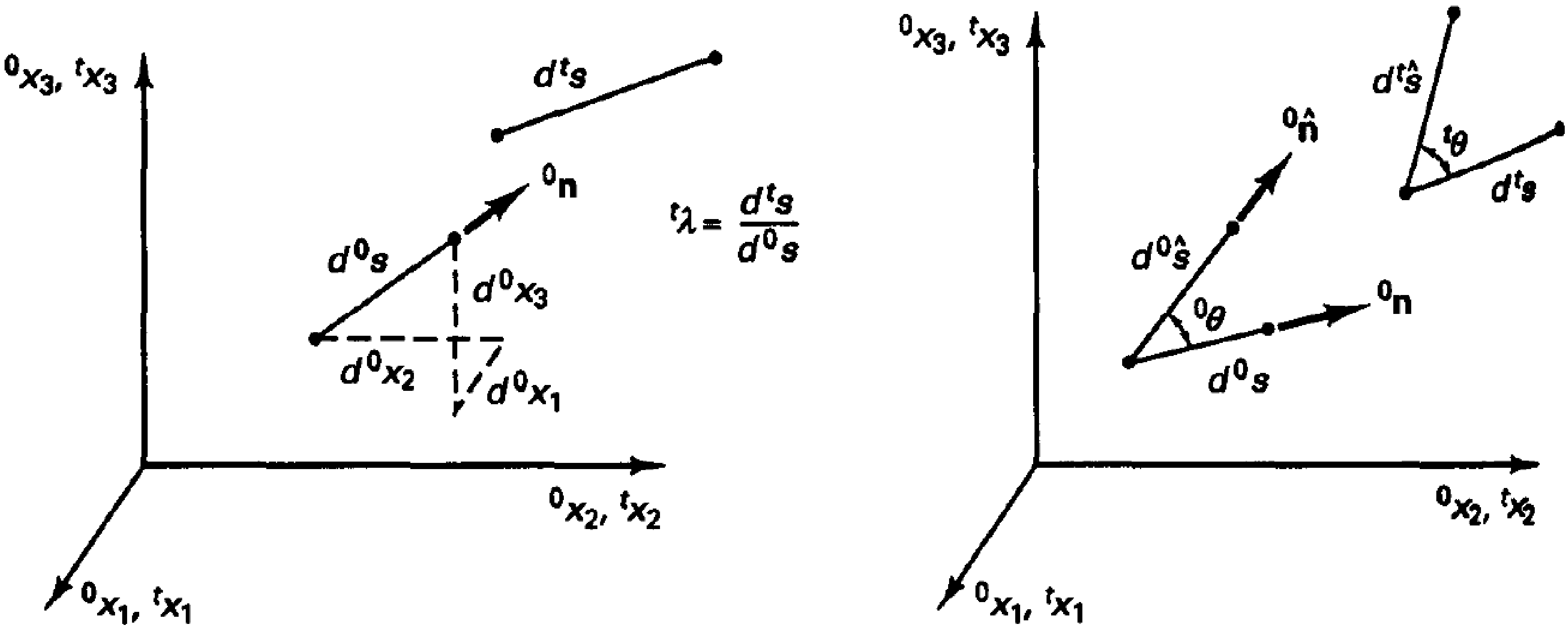


Figure E6.7 Stretch and rotation of line elements

Prove that $\quad {}^{t}\lambda = ({}^{0}\mathbf{n}^{T}\, {}^{t}_{0}\mathbf{C}\, {}^{0}\mathbf{n})^{1/2}$

where ${}^{0}\mathbf{n}$ is a vector of the direction cosines of the line element at time 0. Also, prove that considering two line elements emanating from the same material point, the angle ${}^{t}\theta$ between the line elements at time t is given by

$$\cos {}^{t}\theta = \frac{{}^{0}\mathbf{n}^{T}\, {}^{t}_{0}\mathbf{C}\, {}^{0}\hat{\mathbf{n}}}{{}^{t}\lambda\, {}^{t}\hat{\lambda}} \tag{b}$$

where the hat denotes the second line element (see Fig. E6.7).

To prove (a), we recognize that

$$(d^t s)^2 = d^t\mathbf{x}^T\, d^t\mathbf{x}; \qquad d^t\mathbf{x} = {}_0^t\mathbf{X}\, d^0\mathbf{x}$$

so that using (6.27),
$$(d^t s)^2 = d^0\mathbf{x}^T\, {}_0^t\mathbf{C}\, d^0\mathbf{x}$$

Hence,
$${}^t\lambda^2 = \frac{d^0\mathbf{x}^T}{d^0 s}\, {}_0^t\mathbf{C}\, \frac{d^0\mathbf{x}}{d^0 s}$$

and since
$${}^0\mathbf{n} = \frac{d^0\mathbf{x}}{d^0 s}$$

we have
$${}^t\lambda = ({}^0\mathbf{n}^T\, {}_0^t\mathbf{C}\, {}^0\mathbf{n})^{1/2}$$

To prove (b) we use (2.50)

$$d^t\mathbf{x}^T\, d^t\hat{\mathbf{x}} = (d^t s)(d^t \hat{s}) \cos {}^t\theta$$

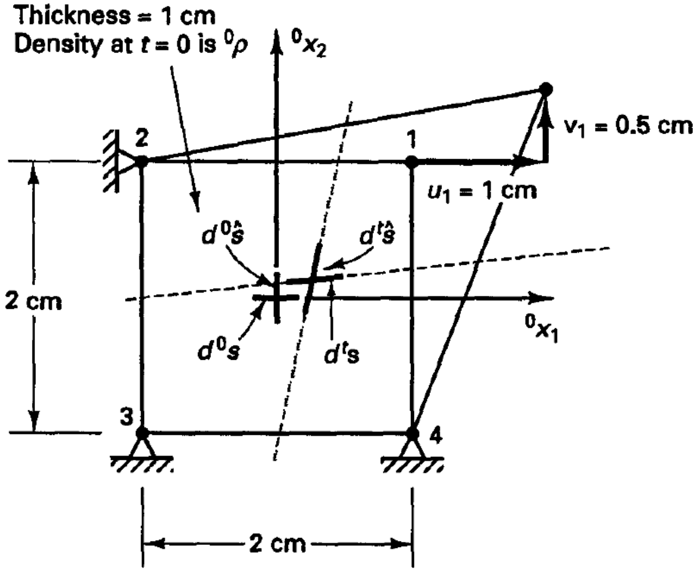
Hence,
$$\cos {}^t\theta = \frac{d^0\mathbf{x}^T\, {}_0^t\mathbf{X}^T\, {}_0^t\hat{\mathbf{X}}\, d^0\hat{\mathbf{x}}}{(d^t s)(d^t \hat{s})}$$

$$\cos {}^t\theta = \frac{{}^0\mathbf{n}^T\, {}_0^t\mathbf{C}\, {}^0\hat{\mathbf{n}}}{{}^t\lambda\, {}^t\hat{\lambda}}$$

If we apply (a) and (b) to the line elements depicted in Fig. E6.6, we obtain at ${}^0x_1 = 0$, ${}^0x_2 = 0$ (see Example 6.6)

$$ {}_0^t\mathbf{C} = \frac{1}{16}\begin{bmatrix} 25.25 & 7.25 \\ 7.25 & 21.25 \end{bmatrix} $$

$$ {}^0\mathbf{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad {}^0\hat{\mathbf{n}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} $$



Hence, using (a), $\quad {}^t\lambda = 1.256; \qquad {}^t\hat{\lambda} = 1.152$

and using (b), $\quad \cos {}^t\theta = 0.313; \qquad {}^t\theta = 71.75^\circ$

Therefore, the angular distortion between the line elements $d\,{}^0s$ and $d\,{}^0\hat{s}$ due to the motion from time 0 to time t is 18.25 degrees.

$$\boxed{{}^t_0\mathbf{X} = {}^t_0\mathbf{R}\;{}^t_0\mathbf{U}}$$

EXAMPLE 6.9: Consider the four-node element and its deformation shown in Fig. E6.9. (a) Evaluate the deformation gradient and its polar decomposition at time t. (b) Assume that the motion from time t to time $t + \Delta t$ consists only of a counterclockwise rigid body rotation of 45 degrees. Evaluate the new deformation gradient.

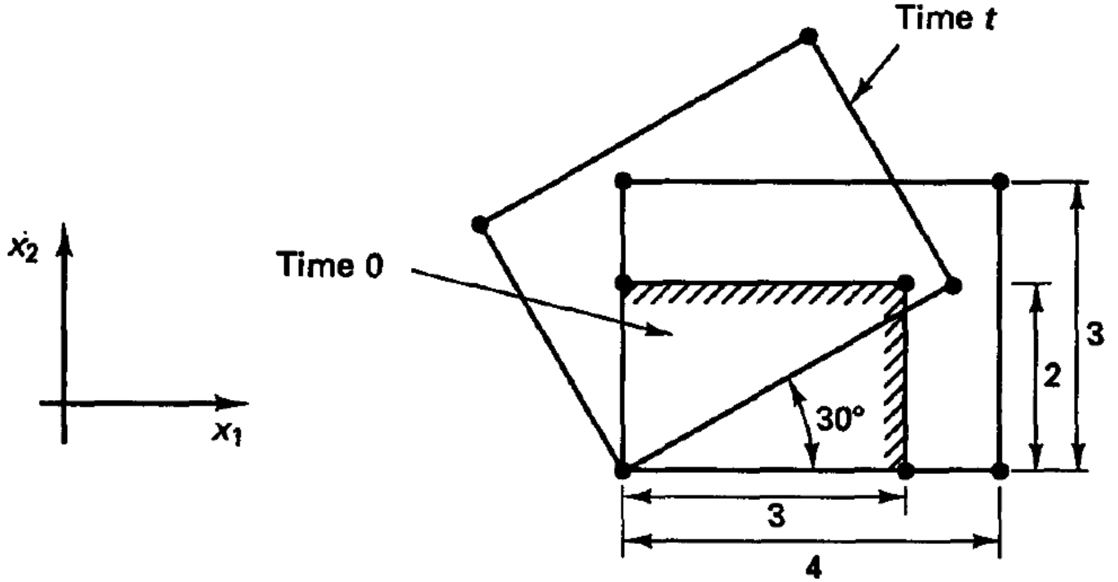


Figure E6.9 Four-node element subjected to stretching and rotation

To evaluate the deformation gradient at time t, we can here conveniently use ${}^t_0\mathbf{X} = {}^t_\tau\mathbf{R}\,{}^\tau_0\mathbf{U}$, where the hypothetical (or conceptual) configuration τ corresponds to the stretching of the fibers only. Hence,

$$
{}^t_\tau\mathbf{R} = \begin{bmatrix} \dfrac{\sqrt{3}}{2} & \dfrac{-1}{2} \\[2ex] \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \end{bmatrix}; \qquad {}^\tau_0\mathbf{U} = \begin{bmatrix} \dfrac{4}{3} & 0 \\[2ex] 0 & \dfrac{3}{2} \end{bmatrix}
$$

and

$$
{}_0^t\mathbf{X} = \begin{bmatrix} \dfrac{2}{\sqrt{3}} & -\dfrac{3}{4} \\ \dfrac{2}{3} & \dfrac{3\sqrt{3}}{4} \end{bmatrix}
$$

Of course, the same result is also obtained by writing ${}^t x_i$ in terms of ${}^0 x_j$, $i = 1, 2$; $j = 1, 2$, and using the definition of ${}_0^t\mathbf{X}$ given in (6.19).

F is one point tensor. So it transforms as follows:

Let us next subject the element to the counterclockwise rotation of 45 degrees. The deformation gradient is then

$$
{}^{t+\Delta t}_{0}\mathbf{X} = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} \dfrac{2}{\sqrt{3}} & -\dfrac{3}{4} \\ \dfrac{2}{3} & \dfrac{3\sqrt{3}}{4} \end{bmatrix}
$$

$$
= \frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{2\sqrt{3} - 2}{3} & -\dfrac{3 + 3\sqrt{3}}{4} \\ \dfrac{2\sqrt{3} + 2}{3} & \dfrac{-3 + 3\sqrt{3}}{4} \end{bmatrix}
$$

$$\mathbf{C} = \mathbf{X}^T\mathbf{X} = \mathbf{U}^2$$

$$\mathbf{X} = \mathbf{V}\mathbf{R}$$

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T$$

$$\mathbf{L} = \left[\frac{\partial^t \dot{u}_i}{\partial^t x_j}\right]$$

$$\mathbf{L} = \dot{\mathbf{X}}\mathbf{X}^{-1}$$

The *velocity gradient* **L** is defined as the gradient of the velocity field with respect to the *current* position ${}^t x_j$ of the material particles,

The symmetric part of **L** is the *velocity strain tensor* **D** (also called the rate-of-deformation tensor or stretching tensor), and the skew-symmetric part is the *spin tensor* **W** (also called the vorticity tensor). Hence,

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \tag{6.41}$$

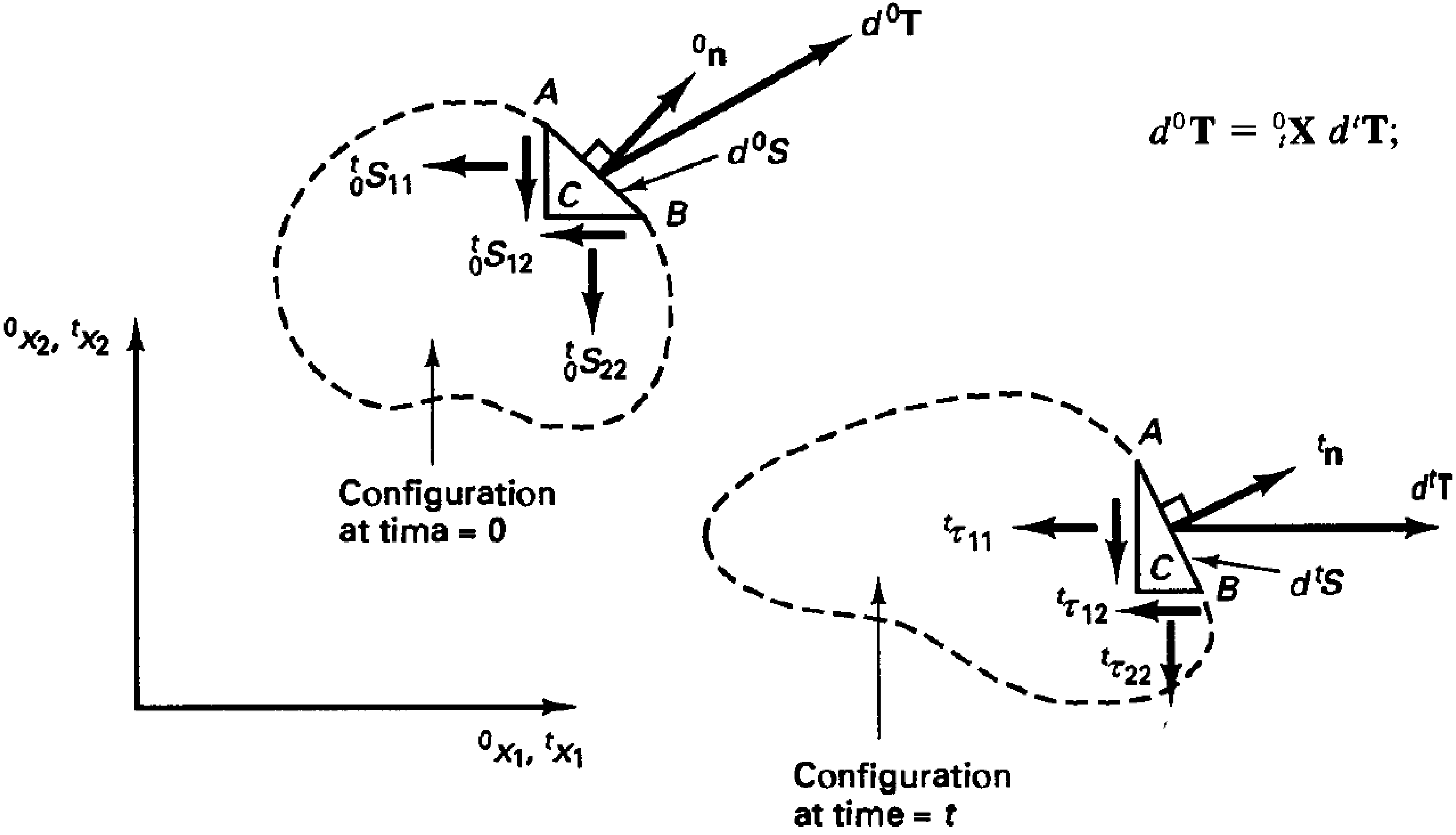
$$\boxed{{}_0^t\dot{\boldsymbol{\epsilon}} = {}_0^t\mathbf{X}^T\, {}^t\mathbf{D}\, {}_0^t\mathbf{X}}$$

$$\boxed{{}^t\mathbf{D} = {}_t^0\mathbf{X}^T\, {}_0^t\dot{\boldsymbol{\epsilon}}\, {}_t^0\mathbf{X}}$$

or in component form (with super- and subscripts)

$$\boxed{{}_0^t\dot{\epsilon}_{ij} = {}_0^t x_{m,i}\, {}_0^t x_{n,j}\, {}^t D_{mn}}$$

$$\boxed{{}^t D_{mn} = {}_t^0 x_{i,m}\, {}_t^0 x_{j,n}\, {}_0^t\dot{\epsilon}_{ij}}$$



$$d^0\mathbf{T} = {}^0_t\mathbf{X}\, d^t\mathbf{T}; \qquad {}^0_t\mathbf{X} = \left[\frac{\partial^0 x_i}{\partial^t x_j}\right]$$

Figure E6.11 Second Piola-Kirchhoff and Cauchy stresses in two-dimensional action

The first Piola-Kirchhoff stress tensor is given by ${}^t_0\mathbf{S}\ {}^t_0\mathbf{X}^T$

However, we shall use the Green-Lagrange strain tensor frequently and now want to define the appropriate stress tensor to use with this strain tensor. The stress measure to use is the *second Piola-Kirchhoff stress* tensor ${}_0^t\mathbf{S}$, which is work-conjugate with the Green-Lagrange strain tensor.[4]

Consider the stress power per unit reference volume ${}^tJ\,{}^t\boldsymbol{\tau}\cdot{}^t\mathbf{D}$,[5] where ${}^t\boldsymbol{\tau}$ is the Cauchy stress tensor and ${}^tJ = \det {}_0^t\mathbf{X}$. Then the second Piola-Kirchhoff stress tensor ${}_0^t\mathbf{S}$ is given by

$$ {}^tJ\,{}^t\boldsymbol{\tau}\cdot{}^t\mathbf{D} = {}_0^t\mathbf{S}\cdot{}_0^t\dot{\boldsymbol{\epsilon}} \tag{6.66} $$

To find the explicit expression for ${}_0^t\mathbf{S}$, we substitute from (6.63) to obtain

$$ {}^tJ\,{}^t\boldsymbol{\tau}\cdot{}^t\mathbf{D} = {}_0^t\mathbf{S}\cdot({}_0^t\mathbf{X}^T\,{}^t\mathbf{D}\,{}_0^t\mathbf{X}) \tag{6.67} $$

Since this relationship must hold for any ${}^t\mathbf{D}$, we have[6]

$$ \boxed{{}_0^t\mathbf{S} = \frac{{}^0\rho}{{}^t\rho}\,{}_t^0\mathbf{X}\,{}^t\boldsymbol{\tau}\,{}_t^0\mathbf{X}^T} \qquad \boxed{{}^t\boldsymbol{\tau} = \frac{{}^t\rho}{{}^0\rho}\,{}_0^t\mathbf{X}\,{}_0^t\mathbf{S}\,{}_0^t\mathbf{X}^T} \tag{6.68} $$

We note that the components of the Green-Lagrange strain tensor and second Piola-Kirchhoff stress tensor do not change when the material is subjected to only a rigid body translation because such motion does not change the deformation gradient.

Example 3.7 An element is deformed through the stages shown in Figure 3.7. The motions between these stages are linear functions of time. Evaluate the rate-of-deformation tensor **D** in each of these stages and obtain the time integral of the rate-of-deformation for the complete cycle of deformation ending in the undeformed configuration.

Each stage of the deformation is assumed to occur over a unit time interval. The time scaling is irrelevant to the results, and we adopt this particular scaling to simplify the algebra. The results would be identical with any other scaling. The motion that takes state 1 to state 2 is

$$x(\mathbf{X}, t) = X + atY, \qquad y(\mathbf{X}, t) = Y \quad 0 \le t \le 1 \tag{E3.7.1}$$

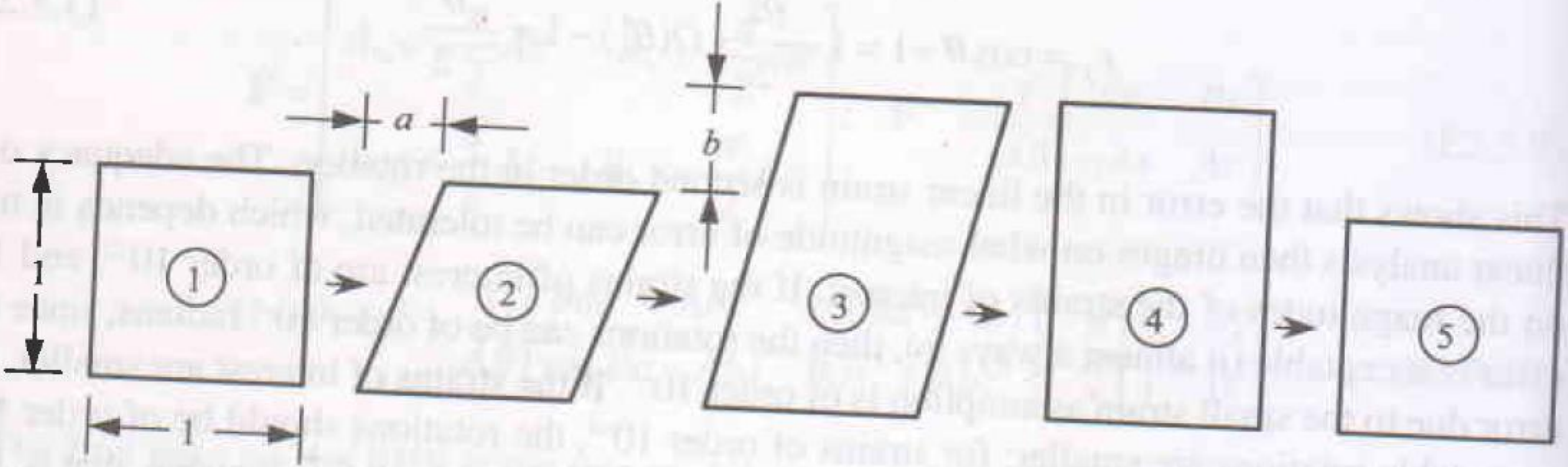


Figure 3.7 An element sheared in the x-direction followed by an extension in the y-direction and then subjected to deformations so that it returns to its initial configuration

To determine the rate-of-deformation, we will use (3.3.18), $\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$, so we first have to determine $\mathbf{F}$, $\dot{\mathbf{F}}$ and $\mathbf{F}^{-1}$. These are

$$\mathbf{F} = \begin{bmatrix} 1 & at \\ 0 & 1 \end{bmatrix}, \quad \dot{\mathbf{F}} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F}^{-1} = \begin{bmatrix} 1 & -at \\ 0 & 1 \end{bmatrix} \tag{E3.7.2}$$

The velocity gradient and rate-of-deformation are then given by (3.3.10):

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -at \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{2} \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \tag{E3.7.3}$$

The Green strain is obtained by (3.3.5):

$$\mathbf{E}=\frac{1}{2}(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{I})=\frac{1}{2}\begin{bmatrix}0 & at\\ at & a^2t^2\end{bmatrix},\quad \dot{\mathbf{E}}=\frac{1}{2}\begin{bmatrix}0 & a\\ a & 2a^2t\end{bmatrix} \tag{E3.7.4}$$

Note that $\dot{E}_{22}$ is nonzero whereas $D_{22}=0$. However, $\dot{E}_{22}$ is small when the constant a is small.

...tion 2 to configuration 3:

$$x(\mathbf{X},t) = X + aY, \quad y(\mathbf{X},t) = (1+bt)Y, \quad 1 \le \bar{t} \le 2, \quad t = \bar{t} - 1 \tag{E3.7.5a}$$

$$\mathbf{F} = \begin{bmatrix} 1 & a \\ 0 & 1+bt \end{bmatrix}, \quad \dot{\mathbf{F}} = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}, \quad \mathbf{F}^{-1} = \frac{1}{1+bt}\begin{bmatrix} 1+bt & -a \\ 0 & 1 \end{bmatrix} \tag{E3.7.5b}$$

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \frac{1}{1+bt}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}, \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{1+bt}\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \tag{E3.7.5c}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) = \frac{1}{2}\begin{bmatrix} 0 & a \\ a & a^2 + bt(bt+2) \end{bmatrix}, \quad \dot{\mathbf{E}} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2b(bt+1) \end{bmatrix} \tag{E3.7.5d}$$

...ation 3 to configuration 4:

$$x(\mathbf{X},t) = X + a(1-t)Y, \quad y(\mathbf{X},t) = (1+b)Y, \quad 2 \le \bar{t} \le 3, \quad t = \bar{t} - 2 \qquad \text{(E3.7.6a)}$$

$$\mathbf{F} = \begin{bmatrix} 1 & a(1-t) \\ 0 & 1+b \end{bmatrix}, \quad \dot{\mathbf{F}} = \begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F}^{-1} = \frac{1}{1+b}\begin{bmatrix} 1+b & a(t-1) \\ 0 & 1 \end{bmatrix} \qquad \text{(E3.7.6b)}$$

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \frac{1}{1+b}\begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \frac{1}{2(1+b)}\begin{bmatrix} 0 & -a \\ -a & 0 \end{bmatrix} \qquad \text{(E3.7.6c)}$$

...uration 4 to configuration 5:*

$$x(\mathbf{X},t) = X, \quad y(\mathbf{X},t) = (1+b-bt)Y, \quad 3 \le \bar{t} \le 4, \quad t = \bar{t} - 3 \qquad \text{(E3.7.7a)}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 0 & 1+b-bt \end{bmatrix}, \quad \dot{\mathbf{F}} = \begin{bmatrix} 0 & 0 \\ 0 & -b \end{bmatrix}, \quad \mathbf{F}^{-1} = \frac{1}{1+b-bt}\begin{bmatrix} 1+b-bt & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(E3.7.7b)}$$

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \frac{1}{1+b-bt}\begin{bmatrix} 0 & 0 \\ 0 & -b \end{bmatrix}, \quad \mathbf{D} = \mathbf{L} \qquad \text{(E3.7.7c)}$$

The Green strain in configuration 5 vanishes, since at $\bar{t} = 4$ the deformation gradient is the unit tensor, $\mathbf{F} = \mathbf{I}$. The time integral of the rate-of-deformation is given by

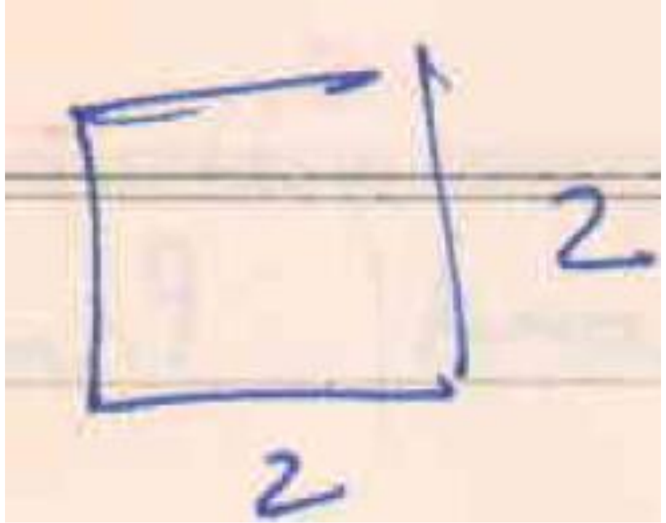
$$\int \mathbf{D}(t)dt = \frac{1}{2}\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \ln(1+b) \end{bmatrix} + \frac{1}{2(1+b)}\begin{bmatrix} 0 & -a \\ -a & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\ln(1+b) \end{bmatrix}$$

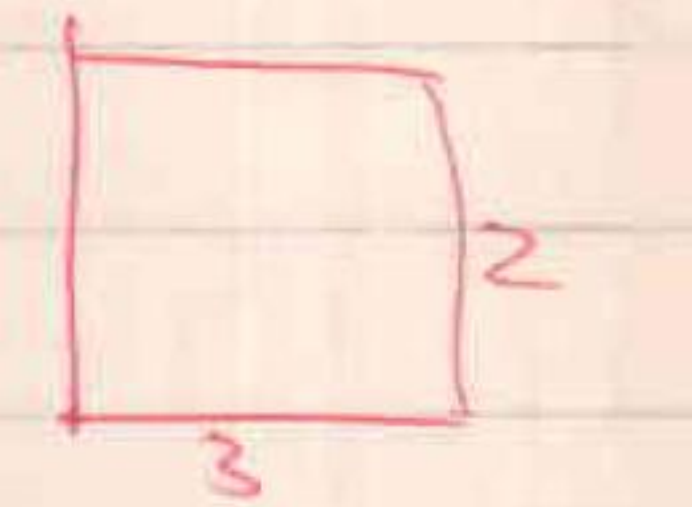
$$= \frac{ab}{2(1+b)}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{(E3.7.8)}$$

- Integral of D is not zero at end of deformation when the body has returned to its original configuration.

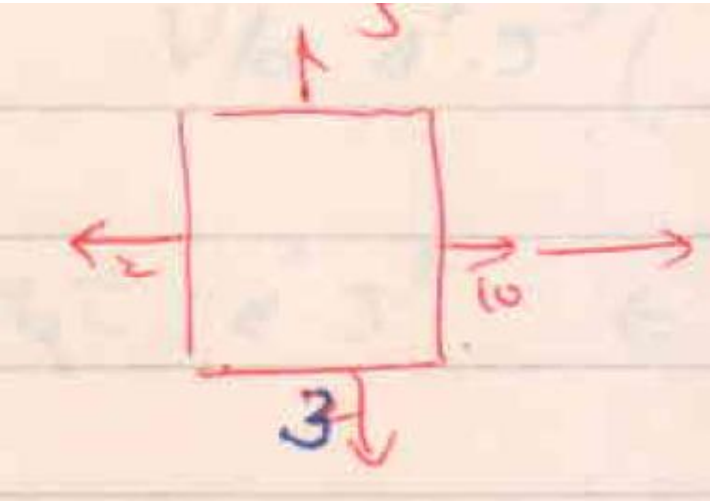
first Piola

<u>Example 7</u> $\underset{\sim}{S}$, $\underset{\sim}{P}$, $\underset{\sim}{T}$





deformed



Cauchy

$$F = \begin{bmatrix} 3/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$$

$${}^{t}_{0}S = \det F \, F^{-1} \sigma F^{t^{-1}} = \frac{3}{2} \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \frac{3}{2} \begin{bmatrix} 2/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2/3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20/3 & 0 \\ 0 & 15/2 \end{bmatrix}$$

$${}^t_0P = (\det F)\,\tau\,(F^T)^{-1}$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & \frac{15}{2} \end{bmatrix}$$

- First Piola is modified according to area on which it is acting. For P11 the area is not changed so it is 10. For P22 the area has changed.
- So s22 x 3= P22 x 2, for the force to be same.

- Now Cauchy shear stress is symmetric, but acting on different faces it will give different force and therefore First Piola stress will be unsymmetric.

- HW 6.4, 6.21,6.23,6.28

Show

$$\int \tau : D \, dV = \int {}^{t}_{0}S : {}^{t}_{0}\epsilon \, d^{0}V$$

Proof:

$$\int \tau_{ij} D_{ij} \, dV$$

$$= \int \tau_{ij} \, (F^{-1})_{ip} \, {}^{t}_{0}\dot{\epsilon}_{pq} (F^{-1})_{qj} \, J \, d^{0}V$$

$$= \int J (F^{-1})_{qj} \, \tau_{ji} (F^{-1})^{T}_{ip} \, {}^{t}_{0}\dot{\epsilon}_{pq} \, d^{0}V$$

$$= \int S_{qp} \, {}^{t}_{0}\dot{\epsilon}_{pq} \, d^{0}V$$

$$= \int S_{pq} \, {}^{t}_{0}\dot{\epsilon}_{pq} \, d^{0}V$$

HW

Starting from

$$\text{Power} = \int \underset{\sim}{t} \cdot \underset{\sim}{v} \, dA + \int \underset{\sim}{f}^{b} \cdot \underset{\sim}{v} \, dV$$

show

$$\text{Power} = \int T_{ij} D_{ij} \, dV$$

$$\int_A \underset{\rightarrow}{t} \cdot \underset{\rightarrow}{v}\, dA = \int_A \tau_{ij}\, n_j\, v_i\, dA$$

$$= \int_V \frac{\partial}{\partial x_j} \left(\tau_{ij}\, v_i \right) dV$$

$$= \int_V \frac{\partial}{\partial x_j} \left(\tau_{ij}\, v_i \right) dV$$

$$= \int_V \left(\frac{\partial}{\partial x_j} \tau_{ij}\, v_i + \tau_{ij} \frac{\partial v_i}{\partial x_j} \right) dV$$

$$\int \underset{\rightarrow}{t} \cdot \underset{\rightarrow}{v}\, dA + \int \underset{\rightarrow}{f}^{b} \cdot \underset{\rightarrow}{v}\, dV$$

$$= \int \left(\frac{\partial}{\partial x_j} \tau_{ij} + f_i^{b} \right) v_i\, dV + \int \tau_{ij} \frac{\partial v_i}{\partial x_j}\, dV$$

$$= \underset{\text{(from equilibrium)}}{0} + \int \tau_{ij}\, L_{ij}\, dV$$

$$= \int_V \tau_{ij} (D_{ij} + W_{ij}) \, dV$$

$$= \int \tau_{ij} D_{ij} \, dV$$ as W_{ij} is ~~sym~~ anti-sym and $\tau_{ij} W_{ij} = 0$

Green Lagrange Strain Tensor

$$ {}^{1}_{0}E_{ij} = \frac{1}{2}\left(\frac{\partial\ {}^{1}x_k}{\partial\ {}^{0}x_i}\frac{\partial\ {}^{1}x_k}{\partial\ {}^{0}x_j} - \delta_{ij}\right) = \frac{1}{2}\left(\frac{\partial\ {}^{1}u_i}{\partial\ {}^{0}x_j} + \frac{\partial\ {}^{1}u_j}{\partial\ {}^{0}x_i} + \frac{\partial\ {}^{1}u_k}{\partial\ {}^{0}x_i}\frac{\partial\ {}^{1}u_k}{\partial\ {}^{0}x_j}\right) $$

$$ {}^{2}_{0}E_{ij} = \frac{1}{2}\left(\frac{\partial\ {}^{2}x_k}{\partial\ {}^{0}x_i}\frac{\partial\ {}^{2}x_k}{\partial\ {}^{0}x_j} - \delta_{ij}\right) = \frac{1}{2}\left(\frac{\partial\ {}^{1}u_i}{\partial\ {}^{0}x_j} + \frac{\partial\ {}^{2}u_j}{\partial\ {}^{0}x_i} + \frac{\partial\ {}^{2}u_k}{\partial\ {}^{0}x_i}\frac{\partial\ {}^{2}u_k}{\partial\ {}^{0}x_j}\right) $$

Green Lagrange Incremental Strain Tensor

$$
\begin{aligned}
2\ {}_{0}\varepsilon_{ij}\mathrm{d}\,{}^{0}x_i\mathrm{d}\,{}^{0}x_j &= ({}^{2}ds)^2 - ({}^{1}ds)^2 \\
&= [({}^{2}ds)^2 - ({}^{0}ds)^2] - [({}^{1}ds)^2 - ({}^{0}ds)^2] \\
&= \left[2\left({}_{0}^{2}E_{ij} - {}_{0}^{1}E_{ij}\right)\right]\mathrm{d}\,{}^{0}x_i\mathrm{d}\,{}^{0}x_j
\end{aligned}
$$

$$2\ {}_0\varepsilon_{ij}\mathrm{d}\ {}^0x_i\mathrm{d}\ {}^0x_j \quad = \left[2\left({}_0^2E_{ij} - {}_0^1E_{ij}\right)\right]\mathrm{d}\ {}^0x_i\mathrm{d}\ {}^0x_j$$

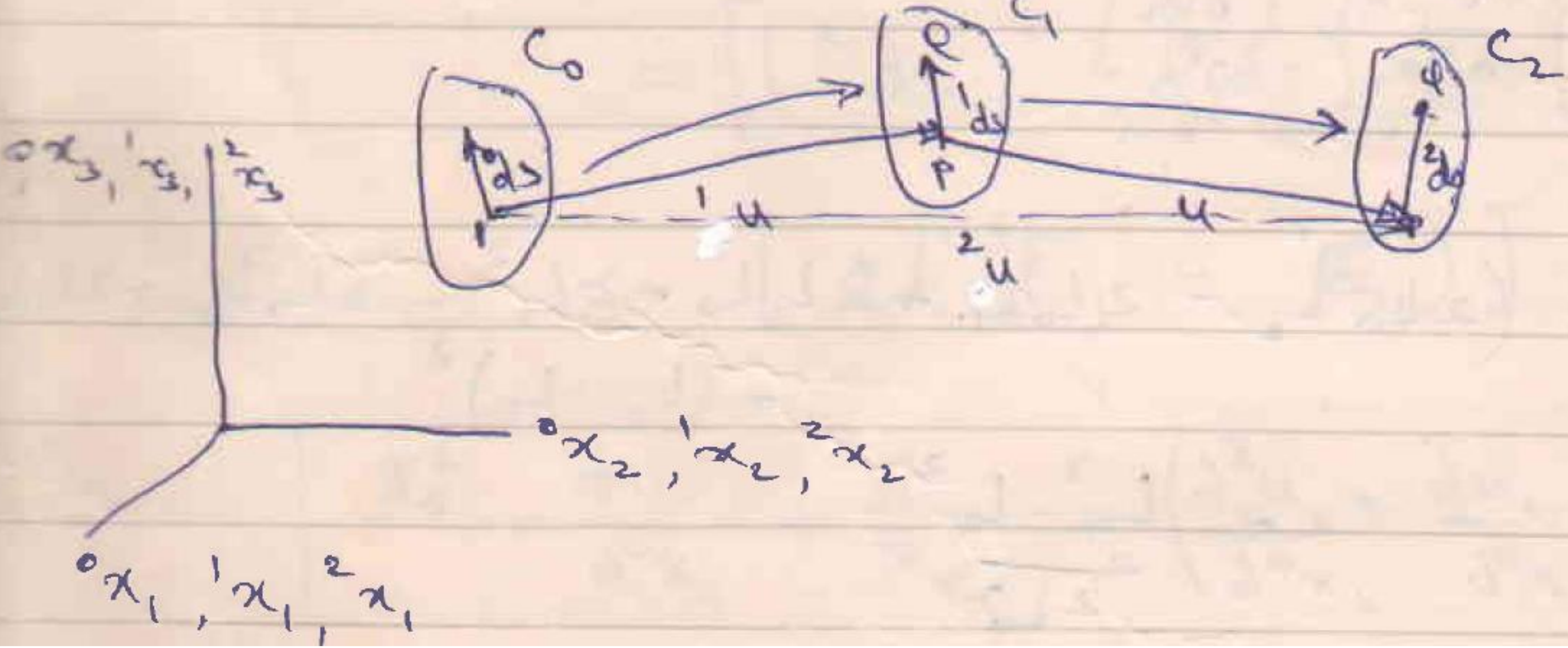
$$=\frac{\partial u_i}{\partial\ {}^0x_j} + \frac{\partial u_j}{\partial\ {}^0x_i} + \left(\frac{\partial\ {}^1u_k}{\partial\ {}^0x_i} + \frac{\partial u_k}{\partial\ {}^0x_i}\right)\left(\frac{\partial\ {}^1u_k}{\partial\ {}^0x_j} + \frac{\partial u_k}{\partial\ {}^0x_j}\right) - \frac{\partial\ {}^1u_k}{\partial\ {}^0x_i}\frac{\partial\ {}^1u_k}{\partial\ {}^0x_j}$$

$$=2\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial\ {}^0x_j} + \frac{\partial u_j}{\partial\ {}^0x_i} + \frac{\partial\ {}^1u_k}{\partial\ {}^0x_i}\frac{\partial u_k}{\partial\ {}^0x_i} + \frac{\partial u_k}{\partial\ {}^0x_i}\frac{\partial\ {}^1u_k}{\partial\ {}^0x_j}\right) + \frac{1}{2}\frac{\partial\ {}^1u_k}{\partial\ {}^0x_i}\frac{\partial^1u_k}{\partial\ {}^0x_j}\right]\mathrm{d}\ {}^0x_i\mathrm{d}\ {}^0x_j$$

$$=2\left(\ {}_0e_{ij} + \ {}_0\eta_{ij}\right)\mathrm{d}\ {}^0x_i\mathrm{d}\ {}^0x_j$$

${}_0e_{ij}$ *is the linear part*

${}_0\eta_{ij}$ *is the non linear part*



C_0
C_1
C_2
Q
Q
P
^0ds
^1ds
^2ds
^1u
^2u
u
$^0x_3, {}^1x_3, {}^2x_3$
$^0x_2, {}^1x_2, {}^2x_2$
$^0x_1, {}^1x_1, {}^2x_1$

- Strain and Stress measures between Configurations
- To determine the final configuration C from a known initial configuration C_0 is to assume that the total load is applied in increments so that the body occupies several intermediate configurations, C_i (i=1,2,3….), prior to occupying the final configuration.
- In determination of an intermediate Configuration C_i , the Lagrangian description of motion can use any of the previously known configurations C_0 , C_1 …… C_{i-1} as reference configuration.

- Total Lagrangian C_0 is reference
- Updated Lagrangian C_{i-1} is reference where C_{i-1} is the latest known reference configuration.

1) C_1 the last known configuration and all variables upto this configuration are known
2) We need to develop a formulation for determining the displacement field of the body in the current deformed configuration C_2

- It is assumed that deformation of the body from C_0 to C_1 due to an increment in load is small and accumulated deformation of body from to can be arbitrarily large but continuous (i.e. neighbor-hoods move into neighbor-hoods)