Newton-Raphson method

DE $$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + \rho B_i = 0$$

$$\delta W_{int} = \int \bar{\epsilon}\,\sigma A\,dx \qquad \sigma = E\epsilon$$

$$K = \int B^T E B A\,dx$$

For the DE $$-\frac{d}{dx}\left(u\frac{du}{dx}\right) = f_0 \qquad 0 < x < 1$$

$$K = \int B^T u B\,dx$$

Example of NR

For the DE $\quad -\frac{d}{dx}\left(u\frac{du}{dx}\right) = f_0 \quad 0<x<1$

subjected to BC

$$\left[u\frac{du}{dx}\right]_{x=0} = \hat{Q} \qquad u(1) = \hat{u}$$

$$K_{ij} = \int_{-1}^{1} \cancel{a_0}^{\,1} \left(N_1u_1 + N_2u_2\right) \frac{\partial N_i}{\partial \xi}\frac{\partial N_j}{\partial \xi}\left(\frac{2}{L_e}\right)\left(\frac{2}{L_e}\right)\frac{L_e}{2}\,d\xi$$

$$K = \frac{u_1 + u_2}{2L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assembled matrix for two elements each of length L

$$\frac{1}{2L} \begin{bmatrix} u_1+u_2 & -(u_1+u_2) & 0 \\ -(u_1+u_2) & u_1+2u_2+u_3 & -(u_2+u_3) \\ 0 & -(u_2+u_3) & (u_2+u_3) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} \end{Bmatrix} + \begin{Bmatrix} \hat{Q} \\ 0 \\ Q_2^{(2)} \end{Bmatrix}$$

Using elimination method and eliminating u_3

$$\frac{1}{2L} \begin{bmatrix} u_1+u_2 & -(u_1+u_2) \\ -(u_1+u_2) & (u_1+2u_2+u_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \end{Bmatrix} + \begin{Bmatrix} \hat{Q} \\ \frac{u_2+u_3}{2L}\hat{u} \end{Bmatrix}$$

As a specific prob.

~~Let~~ $\hat{Q} = 0$, $\hat{u} = \sqrt{2}$ $f_0 = -1$

Length of single element $\boxed{L} = 0.5$

Total length of prob = 1 m

and initial guess consistent with BC (essential)

iteration $u_1^{(0)} = 1.0$ $u_2^{(0)} = 1.0$ $u_3^{(0)} = \sqrt{2}$

$$\phi_i^{n+1} = \phi_i^n + \frac{\partial \phi_i^n}{\partial u_j} \Delta u_j^n = 0$$

+ higher order

$$\phi_i^n = K_{ip}^n u_p^n - (f_e^n)_i$$

$$\phi_i^{n+1} = \left(K_{ip} u_p^n - (f_e^n)_i \right) + \frac{\partial}{\partial u_j} \left(\quad \right) \Delta u_j = 0$$

$$T_{ij} \Delta u_j^{(n)} = \left((F_e)_i^n - K_{ip}^n u_p^n \right)$$

$$u_p^{n+1} = u_p^n + \Delta u_p^{(n)}$$

- Residual on forces

If $\text{conv} = \dfrac{\|R_E^{i+1}\|^2}{1 + \|R_E\|^2} = \sum_{J=1}^{n} \left(R_J^{i+1}\right)^2 \Big/ \left(1 + \sum \left(R_{EJ}\right)^2\right) \leq 0.01$

added to prevent division by zero

If conv

Residual on displacements

$$\frac{\sum_{I=1}^{N} \left| U_I^{(i+1)} - U_I^{(i)} \right|^2}{\sum_{I=1}^{N} \left| U_I^{i+1} \right|^2} \leq \epsilon$$

- In Asghar Bhatti's book
- [K]{u} is written as k(d)
- Fe is written as R_E
- T is written as K_T

7.1.2 Finite Element Solution

For the finite element formulation, we first consider a slightly more general problem:

$$\frac{d}{dx}\left(u^2\frac{du}{dx}\right) + q(x) = 0; \qquad 0 < x < L$$

where $q(x)$ is a given function. The solution domain extends from 0 to an arbitrary length L, which is divided into n elements. An arbitrary element extending from coordinates x_1 to x_2 is shown in Figure 7.2. For generality we assume the possibility of natural boundary

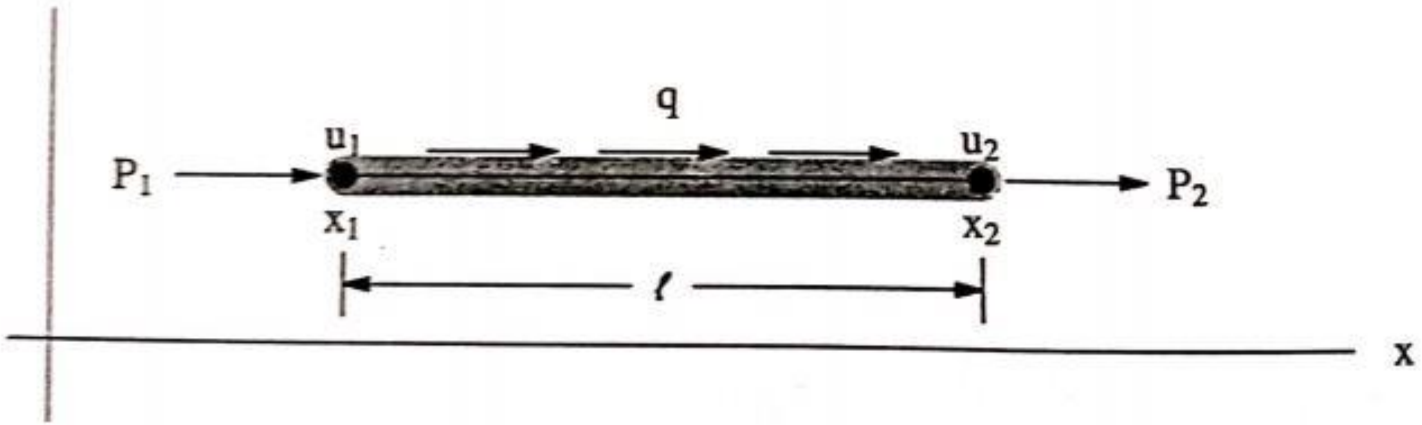

Figure 7.2. Two-node finite element

Tangent element equations for solution using N-R method

$$\boldsymbol{k}(\boldsymbol{d}) = \boldsymbol{r}_E$$

$$\boldsymbol{k}(\boldsymbol{d}) = \int_{x_1}^{x_2} (\boldsymbol{N}^T\boldsymbol{d})^2 \boldsymbol{B}\boldsymbol{B}^T \boldsymbol{d}\, dx = \begin{pmatrix} \dfrac{u_1^3 - u_2^3}{3l} \\ \dfrac{-u_1^3 + u_2^3}{3l} \end{pmatrix}$$

$$\boldsymbol{r}_E = \boldsymbol{r}_q + \boldsymbol{r}_p = \begin{pmatrix} \dfrac{ql}{2} \\ \dfrac{ql}{2} \end{pmatrix} + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

The element tangent matrix is obtained by differentiating these equations with respect to the nodal variables. That is,

$$\boldsymbol{k}_T = \begin{pmatrix} \dfrac{\partial k_1}{\partial u_1} & \dfrac{\partial k_1}{\partial u_2} \\ \dfrac{\partial k_2}{\partial u_1} & \dfrac{\partial k_2}{\partial u_2} \end{pmatrix} = \begin{pmatrix} \dfrac{u_1^2}{l} & -\dfrac{u_2^2}{l} \\ -\dfrac{u_1^2}{l} & \dfrac{u_2^2}{l} \end{pmatrix}$$

Example 7.1 Three-Element Solution of the Problem To illustrate the complete solution process of a nonlinear problem, consider solution of the boundary value problem using three finite elements. The model is shown in Figure 7.3. The load $q = 4$.

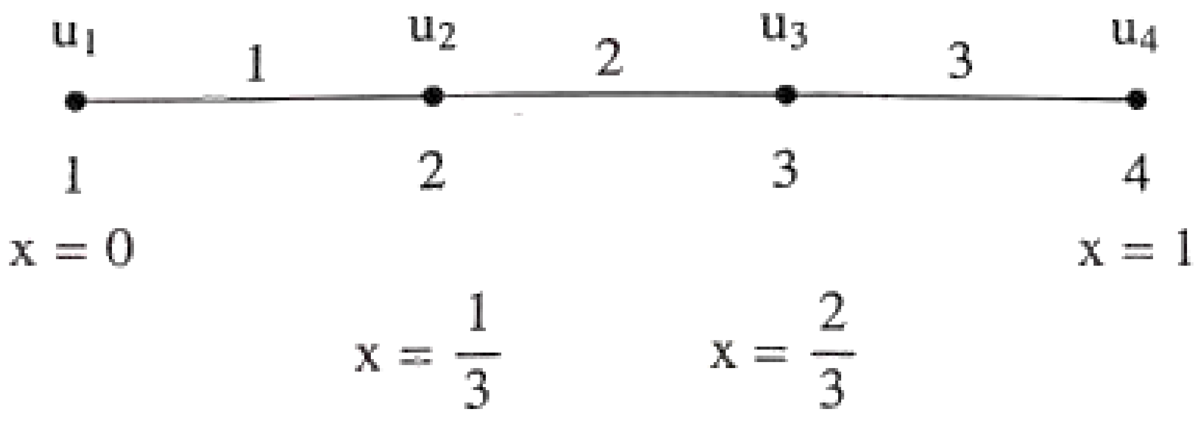

Figure 7.3. Three-element model

Start with following values: $\boldsymbol{d}^{(0)}=\{1,2,2,2\}$

Iteration 1:

$\boldsymbol{d}^{(0)}=\{1,2,2,2\}$

Due to NBC at node 4, $\boldsymbol{R_E=\{0,0,0,2\}}$

Ex. 7.2: To illustrate the process, consider the solution of the nonlinear boundary value problem using three finite elements, shown in fig. 7.3.

One-time calculations:

Element 1: $k_T^{(0)} = \begin{pmatrix} 3 & -12 \\ -3 & 12 \end{pmatrix}$

Element 2: $k_T^{(0)} = \begin{pmatrix} 12 & -12 \\ -12 & 12 \end{pmatrix}$

Element 3: $k_T^{(0)} = \begin{pmatrix} 12 & -12 \\ -12 & 12 \end{pmatrix}$

After assembly:

$$K_T^{(0)} = \begin{pmatrix} 3 & -12 & 0 & 0 \\ -3 & 24 & -12 & 0 \\ 0 & -12 & 24 & -12 \\ 0 & 0 & -12 & 12 \end{pmatrix}$$

Because of EBC, $\{\Delta u_1\} = \{0\}$. After EBC, we have

$$K_T^{(0)} = \begin{pmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{pmatrix}$$

Iteration 1:

$$d^{(0)} = \{1, 2, 2, 2\}$$

Due to NBC at node 4, $R_E = \{0, 0, 0, 2\}$.

Element 1: $r_I^{(0)} = k(d^{(0)}) = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$; $\quad r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$

Element 2: $r_I^{(0)} = k(d^{(0)}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\quad r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$

Element 3: $r_I^{(0)} = k(d^{(0)}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\quad r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$

After assembly:

$$R_I^{(0)} = \begin{pmatrix} -7 \\ 7 \\ 0 \\ 0 \end{pmatrix}; \quad R_E = \begin{pmatrix} 0.666667 \\ 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}$$

Because of EBC, $\{\Delta u_1\} = \{0\}$. After adjusting for EBC, we have

$$R_I^{(0)} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}; \quad R_E = \begin{pmatrix} 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}; \quad R = R_E - R_I^{(0)} = \begin{pmatrix} -5.66667 \\ 1.33333 \\ 2.66667 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|R_E\|^2} = 3.51429$$

Final incremental equations:

$$\begin{pmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} -5.66667 \\ 1.33333 \\ 2.66667 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.138889, 0.194444, 0.416667\}$$

Total nodal values, $d^{(1)} = \{1, 1.86111, 2.19444, 2.41667\}$.

Iteration 2:

$$d^{(1)} = \{1, 1.86111, 2.19444, 2.41667\}$$

Due to NBC at node 4, $R_E = \{0, 0, 0, 2\}$.

Element 1: $r_I^{(1)} = k(d^{(0)}) = \begin{pmatrix} -5.44639 \\ 5.44639 \end{pmatrix}; \qquad r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$

Element 2: $r_I^{(1)} = k(d^{(0)}) = \begin{pmatrix} -4.12114 \\ 4.12114 \end{pmatrix}; \qquad r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$

Element 3: $r_I^{(1)} = k(d^{(0)}) = \begin{pmatrix} -3.54647 \\ 3.54647 \end{pmatrix}; \qquad r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$

After assembly:

$$R_I^{(1)} = \begin{pmatrix} -5.44639 \\ 1.32525 \\ 0.574674 \\ 3.54647 \end{pmatrix}; \qquad R_E = \begin{pmatrix} 0.666667 \\ 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}$$

Because of EBC, $\{\Delta u_1\} = \{0\}$. After adjusting for EBC, we have

$$R_I^{(0)} = \begin{pmatrix} 1.32525 \\ 0.574674 \\ 3.54647 \end{pmatrix}; \qquad R_E = \begin{pmatrix} 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}; \qquad R = R_E - R_I^{(0)} = \begin{pmatrix} 0.00808042 \\ 0.758659 \\ -0.879801 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|R_E\|^2} = 0.115687$$

Final incremental equations:

$$\begin{pmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} 0.00808042 \\ 0.758659 \\ -0.879801 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.0094218, -0.019517, -0.0928337\}$$

Total nodal values, $d^{(2)} = \{1, 1.85169, 2.17493, 2.32383\}$.

Iteration 3:

After adjusting for EBC:

$$R_I^{(2)} = \begin{pmatrix} 1.40989 \\ 1.67801 \\ 2.26108 \end{pmatrix}; \qquad R_E = \begin{pmatrix} 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}; \qquad R = R_E - R_I^{(2)} = \begin{pmatrix} -0.0765582 \\ -0.34468 \\ 0.405585 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|R_E\|^2} = 0.0247855$$

Final incremental equations:

$$\begin{pmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} -0.0765582 \\ -0.34468 \\ 0.405585 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.00130438, 0.00377109, 0.0375699\}$$

Total nodal values, $d^{(3)} = \{1, 1.85038, 2.1787, 2.3614\}$.

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.0108805, -0.0162115, -0.0635817\}$$

Total nodal values, $d^{(2)} = \{1, 1.85023, 2.17823, 2.35308\}$.

Iteration 3:

$$d^{(2)} = \{1, 1.85023, 2.17823, 2.35308\}$$

Due to NBC at node 4, $R_E = \{0, 0, 0, 2\}$.

Element 1: $k_T^{(2)} = \begin{pmatrix} 3 & -10.2701 \\ -3 & 10.2701 \end{pmatrix}$; $\quad r_I^{(2)} = k(d^{(2)}) = \begin{pmatrix} -5.33399 \\ 5.33399 \end{pmatrix}$;

$$r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$$

Element 2: $k_T^{(2)} = \begin{pmatrix} 10.2701 & -14.2341 \\ -10.2701 & 14.2341 \end{pmatrix}$; $\quad r_I^{(2)} = k(d^{(2)}) = \begin{pmatrix} -4.00107 \\ 4.00107 \end{pmatrix}$;

$$r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$$

Element 3: $k_T^{(2)} = \begin{pmatrix} 14.2341 & -16.611 \\ -14.2341 & 16.611 \end{pmatrix}$; $\quad r_I^{(2)} = k(d^{(2)}) = \begin{pmatrix} -2.69399 \\ 2.69399 \end{pmatrix}$;

$$r_E = \begin{pmatrix} 0.666667 \\ 0.666667 \end{pmatrix}$$

After assembly, the global matrices are:

$$K_T^{(2)} = \begin{pmatrix} 3 & -10.2701 & 0 & 0 \\ -3 & 20.5401 & -14.2341 & 0 \\ 0 & -10.2701 & 28.4682 & -16.611 \\ 0 & 0 & -14.2341 & 16.611 \end{pmatrix};$$

$$R_I^{(2)} = \begin{pmatrix} -5.33399 \\ 1.33293 \\ 1.30707 \\ 2.69399 \end{pmatrix}; \qquad R_E = \begin{pmatrix} 0.666667 \\ 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}$$

Because of EBC, $\{\Delta u_1\} = \{0\}$. After adjusting for EBC, we have

$$K_T^{(2)} = \begin{pmatrix} 20.5401 & -14.2341 & 0 \\ -10.2701 & 28.4682 & -16.611 \\ 0 & -14.2341 & 16.611 \end{pmatrix}; \qquad R_I = \begin{pmatrix} 1.33293 \\ 1.30707 \\ 2.69399 \end{pmatrix}$$

$$R_E = \begin{pmatrix} 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}; \qquad R^{(2)} = R_E - R_I^{(2)} = \begin{pmatrix} 0.000406527 \\ 0.0262599 \\ -0.0273261 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|R_E\|^2} = 0.000123126$$

Complete incremental equations:

$$\begin{pmatrix} 20.5401 & -14.2341 & 0 \\ -10.2701 & 28.4682 & -16.611 \\ 0 & -14.2341 & 16.611 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} 0.000406527 \\ 0.0262599 \\ -0.0273261 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.000064235, -0.000121253, -0.00174896\}$$

Total nodal values, $d^{(3)} = \{1, 1.85017, 2.17811, 2.35134\}$.

The solution converged to the desired tolerance of 0.01. This solution compares reasonably well with the exact solution shown in Figure 7.4. It can obviously be improved further by using more elements.

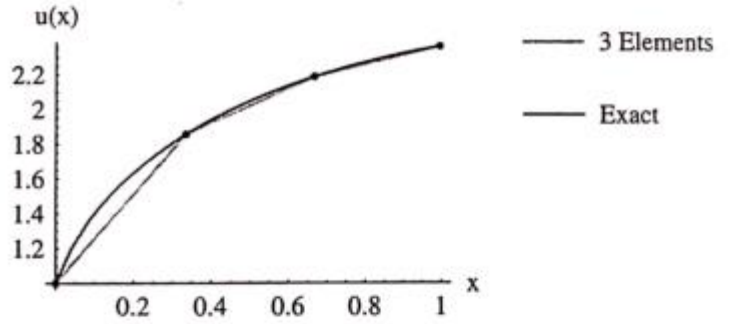

Figure 7.4. Comparison of the three-element solution and exact solution

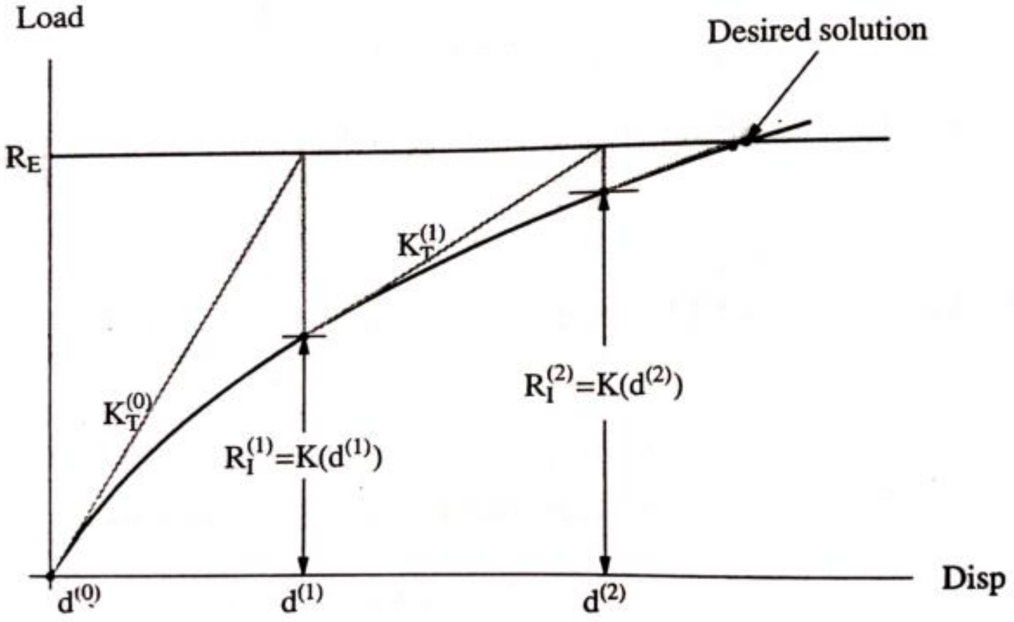

Figure 7.5. Newton-Raphson iteration

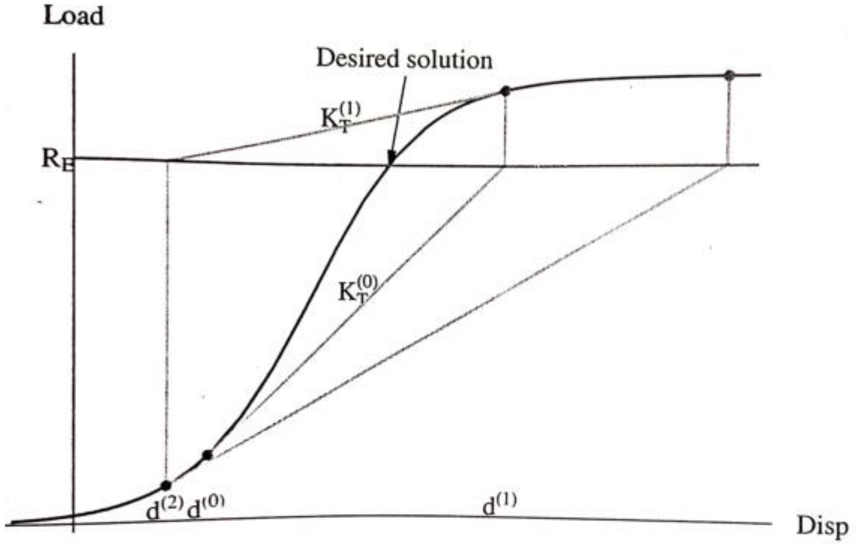

Figure 7.6. Divergence in the Newton-Raphson iteration

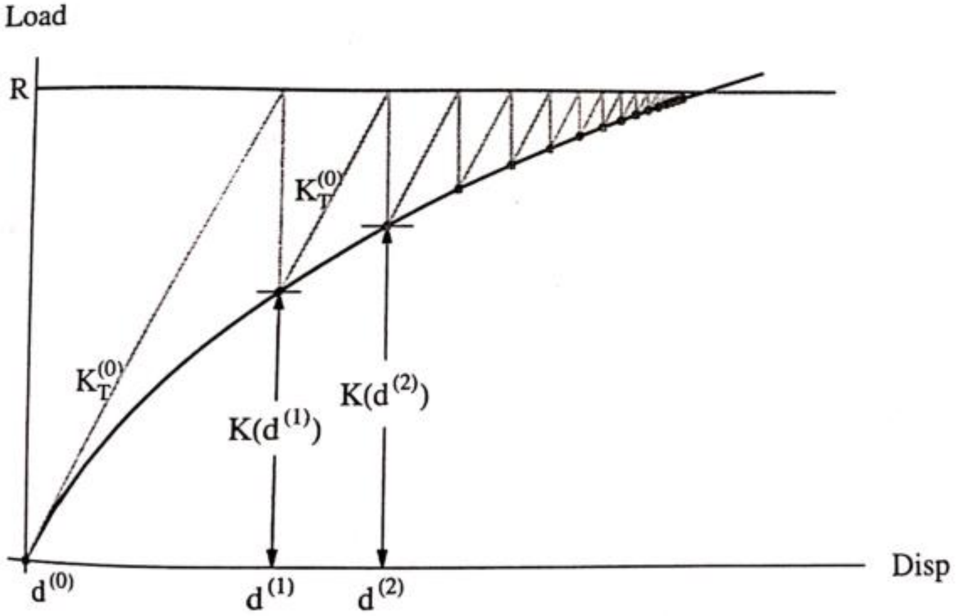

Figure 7.7. Constant-stiffness iteration

Iteration 4:

After adjusting for EBC, we have

$$R_I^{(3)} = \begin{pmatrix} 1.32947 \\ 1.18009 \\ 2.82602 \end{pmatrix}; \qquad R_E = \begin{pmatrix} 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}; \qquad R = R_E - R_I^{(3)} = \begin{pmatrix} 0.0038654 \\ 0.153244 \\ -0.159354 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|R_E\|^2} = 0.00419078$$

Final incremental equations:

$$\begin{pmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} 0.0038654 \\ 0.153244 \\ -0.159354 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.000187068, -0.000696253, -0.0139758\}$$

Total nodal values, $d^{(4)} = \{1, 1.8502, 2.178, 2.34743\}$.

The solution converged to the desired tolerance of 0.01.

Ex. 7.3: To use the load steps, the differential equation and the natural boundary condition are written as follows.

Differential equation: $\frac{d}{dx}\left(u^2\frac{du}{dx}\right) + 4 = 0 \Longrightarrow \frac{d}{dx}\left(u^2\frac{du}{dx}\right) + \lambda q = 0$

NBC: $\left(u^2\frac{du}{dx}\right)_{x=1} = 2 \Longrightarrow \left(u^2\frac{du}{dx}\right)_{x=1} = \lambda p$

If we set p=1/2 and q= 1/2 ,then with we should get the same solution as before

Current Load Parameter, $\lambda = 2$

Nominal Values: NBC at node 4 = ½; q =1

Starting nodal values ={1 , 2 ,2 , 2}

ITERATION 1 :

$$d^{(0)} = \{1, 2, 2, 2\}$$

With the nominal NBC value at node 4, $R_E = \{0, 0, 0, \frac{1}{2}\}$.

Element 1: $k_T^{(0)} = \begin{pmatrix} 3 & -12 \\ -3 & 12 \end{pmatrix}$; $\quad r_I^{(0)} = k(d^{(0)}) = \begin{pmatrix} -7 \\ 7 \end{pmatrix}$; $\quad r_E = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$

Element 2: $k_T^{(0)} = \begin{pmatrix} 12 & -12 \\ -12 & 12 \end{pmatrix}$; $\quad r_I^{(0)} = k(d^{(0)}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\quad r_E = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$

Element 3: $k_T^{(0)} = \begin{pmatrix} 12 & -12 \\ -12 & 12 \end{pmatrix}$; $\quad r_I^{(0)} = k(d^{(0)}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\quad r_E = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$

After assembly, the global matrices are

$$K_T^{(1)} = \begin{pmatrix} 3 & -12 & 0 & 0 \\ -3 & 24 & -12 & 0 \\ 0 & -12 & 24 & -12 \\ 0 & 0 & -12 & 12 \end{pmatrix}; \quad R_I^{(0)} = \begin{pmatrix} -7 \\ 7 \\ 0 \\ 0 \end{pmatrix}; \quad R_E = \begin{pmatrix} 0.166667 \\ 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}$$

Because of EBC, $\{\Delta u_1\} = \{0\}$. After adjusting for EBC, we have

$$K_T^{(1)} = \begin{pmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{pmatrix}; \qquad R_I^{(0)} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix};$$

$$R_E = \begin{pmatrix} 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}; \qquad \lambda R_E = \begin{pmatrix} 0.666667 \\ 0.666667 \\ 1.33333 \end{pmatrix}; \qquad R = \lambda R_E - R_I^{(0)} = \begin{pmatrix} -6.33333 \\ 0.666667 \\ 1.33333 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|\lambda R_E\|^2} = 11.5455$$

Complete incremental equations:

$$\begin{pmatrix} 24 & -12 & 0 \\ -12 & 24 & -12 \\ 0 & -12 & 12 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} -6.33333 \\ 0.666667 \\ 1.33333 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.361111, -0.194444, -0.0833333\}$$

Total nodal values, $\boldsymbol{d}^{(1)} = \{1, 1.63889, 1.80556, 1.91667\}$.

Iteration 2:

$$\boldsymbol{d}^{(1)} = \{1, 1.63889, 1.80556, 1.91667\}$$

With the nominal NBC value at node 4, $\boldsymbol{R}_E = \left\{0, 0, 0, \frac{1}{2}\right\}$.

Element 1: $$\boldsymbol{k}_T^{(1)} = \begin{pmatrix} 3 & -8.05787 \\ -3 & 8.05787 \end{pmatrix}; \qquad \boldsymbol{r}_I^{(1)} = \boldsymbol{k}(\boldsymbol{d}^{(1)}) = \begin{pmatrix} -3.40198 \\ 3.40198 \end{pmatrix};$$

$$\boldsymbol{r}_E = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$$

Element 2: $$\boldsymbol{k}_T^{(1)} = \begin{pmatrix} 8.05787 & -9.78009 \\ -8.05787 & 9.78009 \end{pmatrix}; \qquad \boldsymbol{r}_I^{(1)} = \boldsymbol{k}(\boldsymbol{d}^{(1)}) = \begin{pmatrix} -1.48418 \\ 1.48418 \end{pmatrix};$$

$$\boldsymbol{r}_E = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$$

Element 3: $$\boldsymbol{k}_T^{(1)} = \begin{pmatrix} 9.78009 & -11.0208 \\ -9.78009 & 11.0208 \end{pmatrix}; \qquad \boldsymbol{r}_I^{(1)} = k(\boldsymbol{d}^{(1)}) = \begin{pmatrix} -1.15492 \\ 1.15492 \end{pmatrix};$$

$$\boldsymbol{r}_E = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$$

After assembly, the global matrices are

$$K_T^{(1)} = \begin{pmatrix} 3 & -8.05787 & 0 & 0 \\ -3 & 16.1157 & -9.78009 & 0 \\ 0 & -8.05787 & 19.5602 & -11.0208 \\ 0 & 0 & -9.78009 & 11.0208 \end{pmatrix};$$

$$R_I^{(1)} = \begin{pmatrix} -3.40198 \\ 1.9178 \\ 0.329261 \\ 1.15492 \end{pmatrix}; \qquad R_E = \begin{pmatrix} 0.166667 \\ 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}$$

Because of EBC, $\{\Delta u_1\} = \{0\}$. After adjusting for EBC, we have

$$K_T^{(1)} = \begin{pmatrix} 16.1157 & -9.78009 & 0 \\ -8.05787 & 19.5602 & -11.0208 \\ 0 & -9.78009 & 11.0208 \end{pmatrix}; \qquad R_I^{(1)} = \begin{pmatrix} 1.9178 \\ 0.329261 \\ 1.15492 \end{pmatrix};$$

$$R_E = \begin{pmatrix} 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}; \qquad \lambda R_E = \begin{pmatrix} 0.666667 \\ 0.666667 \\ 1.33333 \end{pmatrix}; \qquad R = \lambda R_E - R_I^{(1)} = \begin{pmatrix} -1.25114 \\ 0.337406 \\ 0.178412 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|\lambda R_E\|^2} = 0.46664$$

Complete incremental equations:

Complete incremental equations:

$$\begin{pmatrix} 16.1157 & -9.78009 & 0 \\ -8.05787 & 19.5602 & -11.0208 \\ 0 & -9.78009 & 11.0208 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} -1.25114 \\ 0.337406 \\ 0.178412 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.0912546, -0.0224436, -0.00372821\}$$

Total nodal values, $\boldsymbol{d}^{(2)} = \{1, 1.54763, 1.78311, 1.91294\}$.

Iteration 3:

After adjusting for EBC, we have

$$K_T^{(2)} = \begin{pmatrix} 14.371 & -9.53847 & 0 \\ -7.18552 & 19.0769 & -10.978 \\ 0 & -9.53847 & 10.978 \end{pmatrix}; \qquad R_I^{(2)} = \begin{pmatrix} 0.744316 \\ 0.631838 \\ 1.3307 \end{pmatrix};$$

$$R_E = \begin{pmatrix} 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}; \qquad \lambda R_E = \begin{pmatrix} 0.666667 \\ 0.666667 \\ 1.33333 \end{pmatrix}; \qquad R = \lambda R_E - R_I^{(2)} = \begin{pmatrix} -0.0776492 \\ 0.0348288 \\ 0.00263727 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|\lambda R_E\|^2} = 0.00197711$$

Solution converged at load step, $\lambda = 2$.
Nodal values, $\boldsymbol{d} = \{1, 1.54763, 1.78311, 1.91294\}$.

Current Load Parameter, $\lambda = 4$

$$\text{Nominal values: NBC at node } 4 = \tfrac{1}{2}; \qquad q = 1$$

$$\text{Starting nodal values} = \{1, 1.54763, 1.78311, 1.91294\}$$

Iteration 1:

$$\boldsymbol{d}^{(0)} = \{1, 1.54763, 1.78311, 1.91294\}$$

With the nominal NBC value at node 4, $\boldsymbol{R}_E = \left\{0, 0, 0, \tfrac{1}{2}\right\}$.

$$\text{Element 1:} \quad \boldsymbol{k}_T^{(0)} = \begin{pmatrix} 3 & -7.18552 \\ -3 & 7.18552 \end{pmatrix}; \qquad \boldsymbol{r}_I^{(0)} = \boldsymbol{k}(\boldsymbol{d}^{(0)}) = \begin{pmatrix} -2.70685 \\ 2.70685 \end{pmatrix};$$

Element 2: $k_T^{(0)} = \begin{pmatrix} 7.18552 & -9.53847 \\ -7.18552 & 9.53847 \end{pmatrix}$; $\quad r_I^{(0)} = k(d^{(0)}) = \begin{pmatrix} -1.96253 \\ 1.96253 \end{pmatrix}$;

$$r_E = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$$

Element 3: $k_T^{(0)} = \begin{pmatrix} 9.53847 & -10.978 \\ -9.53847 & 10.978 \end{pmatrix}$; $\quad r_I^{(0)} = k(d^{(0)}) = \begin{pmatrix} -1.3307 \\ 1.3307 \end{pmatrix}$;

$$r_E = \begin{pmatrix} 0.166667 \\ 0.166667 \end{pmatrix}$$

After assembly, the global matrices are

$$K_T^{(1)} = \begin{pmatrix} 3 & -7.18552 & 0 & 0 \\ -3 & 14.371 & -9.53847 & 0 \\ 0 & -7.18552 & 19.0769 & -10.978 \\ 0 & 0 & -9.53847 & 10.978 \end{pmatrix};$$

$$\boldsymbol{R}_I^{(0)} = \begin{pmatrix} -2.70685 \\ 0.744316 \\ 0.631838 \\ 1.3307 \end{pmatrix}; \qquad \boldsymbol{R}_E = \begin{pmatrix} 0.166667 \\ 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}$$

Because of EBC, $\{\Delta u_1\} = \{0\}$. After adjusting for EBC, we have

$$\boldsymbol{K}_T^{(0)} = \begin{pmatrix} 14.371 & -9.53847 & 0 \\ -7.18552 & 19.0769 & -10.978 \\ 0 & -9.53847 & 10.978 \end{pmatrix}; \qquad \boldsymbol{R}_I^{(0)} = \begin{pmatrix} 0.744316 \\ 0.631838 \\ 1.3307 \end{pmatrix};$$

$$\boldsymbol{R}_E = \begin{pmatrix} 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}; \qquad \lambda \boldsymbol{R}_E = \begin{pmatrix} 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}; \qquad \boldsymbol{R} = \lambda \boldsymbol{R}_E - \boldsymbol{R}_I^{(0)} = \begin{pmatrix} 0.589017 \\ 0.701495 \\ 1.33597 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|\boldsymbol{R}\|^2}{1 + \|\lambda \boldsymbol{R}_E\|^2} = 0.228555$$

Complete incremental equations:

$$\begin{pmatrix} 14.371 & -9.53847 & 0 \\ -7.18552 & 19.0769 & -10.978 \\ 0 & -9.53847 & 10.978 \end{pmatrix} \begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} 0.589017 \\ 0.701495 \\ 1.33597 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{0.365525, 0.488962, 0.54654\}$$

Total nodal values, $\boldsymbol{d}^{(1)} = \{1, 1.91316, 2.27207, 2.45948\}$.

Iteration 2:

After adjusting for EBC, we have

$$K_T^{(1)} = \begin{pmatrix} 21.9611 & -15.487 & 0 \\ -10.9805 & 30.9739 & -18.1471 \\ 0 & -15.487 & 18.1471 \end{pmatrix}; \qquad R_I^{(1)} = \begin{pmatrix} 1.27582 \\ 1.57838 \\ 3.1483 \end{pmatrix};$$

$$R_E = \begin{pmatrix} 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}; \qquad \lambda R_E = \begin{pmatrix} 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}; \qquad R = \lambda R_E - R_I^{(0)} = \begin{pmatrix} 0.0575091 \\ -0.245045 \\ -0.481631 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|\lambda R_E\|^2} = 0.023853$$

Complete incremental equations:

$$\begin{pmatrix} 21.9611 & -15.487 & 0 \\ -10.9805 & 30.9739 & -18.1471 \\ 0 & -15.487 & 18.1471 \end{pmatrix}\begin{pmatrix} \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{pmatrix} = \begin{pmatrix} 0.0575091 \\ -0.245045 \\ -0.481631 \end{pmatrix}$$

Solution:

$$\{\Delta u_2, \Delta u_3, \Delta u_4\} = \{-0.0609413, -0.0901303, -0.103459\}$$

Total nodal values, $\boldsymbol{d}^{(2)} = \{1, 1.85222, 2.18194, 2.35602\}$.

Iteration 3:

After adjusting for EBC, we have

$$\boldsymbol{K}_T^{(2)} = \begin{pmatrix} 20.5843 & -14.2826 & 0 \\ -10.2921 & 28.5653 & -16.6525 \\ 0 & -14.2826 & 16.6525 \end{pmatrix}; \qquad \boldsymbol{R}_I^{(2)} = \begin{pmatrix} 1.32087 \\ 1.34365 \\ 2.6899 \end{pmatrix};$$

$$R_E = \begin{pmatrix} 0.333333 \\ 0.333333 \\ 0.666667 \end{pmatrix}; \qquad \lambda R_E = \begin{pmatrix} 1.33333 \\ 1.33333 \\ 2.66667 \end{pmatrix}; \qquad R = \lambda R_E - R_I^{(2)} = \begin{pmatrix} 0.0124608 \\ -0.01032 \\ -0.02323 \end{pmatrix}$$

Convergence parameter:

$$\frac{\|R\|^2}{1 + \|\lambda R_E\|^2} = 0.0000593649$$