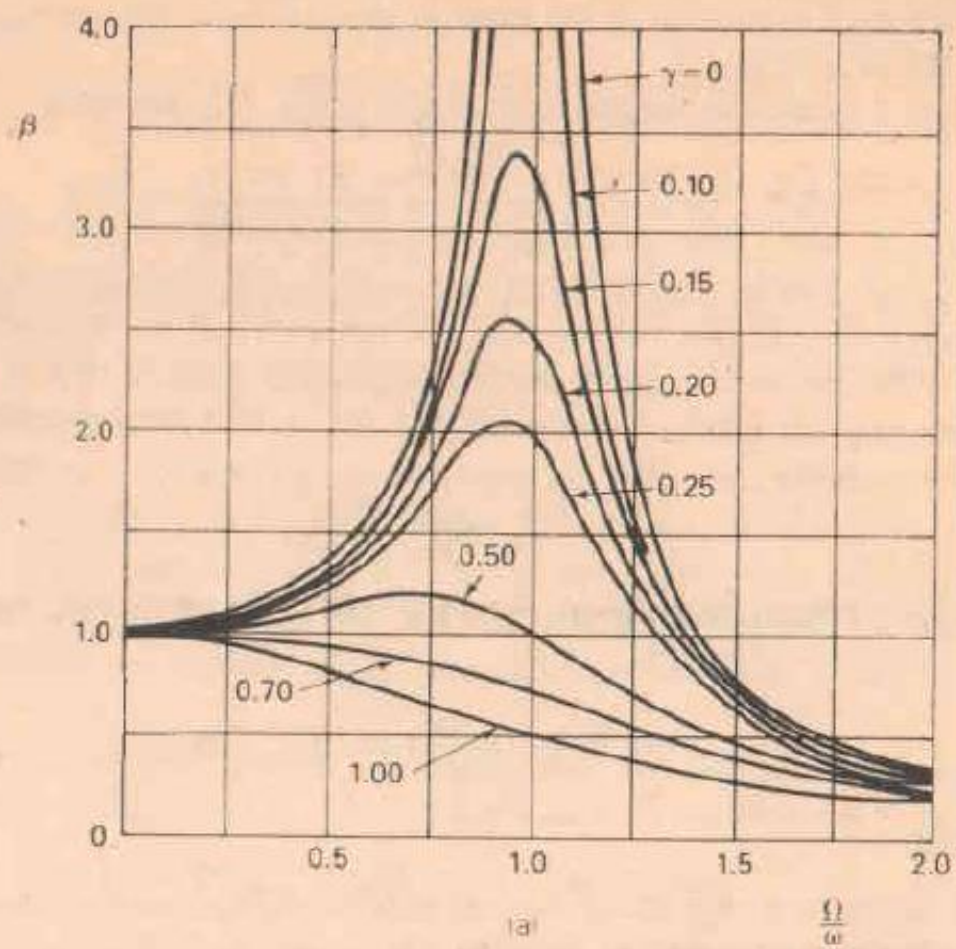


# Dynamics



- $\beta$  is the magnification factor i.e ratio of dynamic to static response
- $\omega$  is the natural frequency whereas capital omega is the forcing frequency,

$$\mathbf{K}\mathbf{U} = \mathbf{R}$$

$$\mathbf{R} = \mathbf{R}_B + \mathbf{R}_S - \mathbf{R}_I + \mathbf{R}_C$$

$$\mathbf{K} = \sum_m \underbrace{\int_{V^{(m)}} \mathbf{B}^{(m)T} \mathbf{C}^{(m)} \mathbf{B}^{(m)} dV^{(m)}}_{= \mathbf{K}^{(m)}}$$

$$\mathbf{R}_S = \sum_m \underbrace{\int_{S_1^{(m)}, \dots, S_q^{(m)}} \mathbf{H}^{S(m)T} \mathbf{f}^{S(m)} dS^{(m)}}_{= \mathbf{R}_S^{(m)}}$$

$$\mathbf{R}_B = \sum_m \underbrace{\int_{V^{(m)}} \mathbf{H}^{(m)T} \mathbf{f}^{B(m)} dV^{(m)}}_{= \mathbf{R}_B^{(m)}}$$

$$\mathbf{R}_I = \sum_m \underbrace{\int_{V^{(m)}} \mathbf{B}^{(m)T} \boldsymbol{\tau}^{I(m)} dV^{(m)}}_{= \mathbf{R}_I^{(m)}}$$

$$\mathbf{R}_B = \sum_m \int_{V^{(m)}} \mathbf{H}^{(m)T} [\mathbf{f}^{B(m)} - \rho^{(m)} \mathbf{H}^{(m)} \ddot{\mathbf{U}}] \, dV^{(m)}$$

$$\mathbf{M} \ddot{\mathbf{U}} + \mathbf{K} \mathbf{U} = \mathbf{R}$$

$$\mathbf{M} = \sum_m \underbrace{\int_{V^{(m)}} \rho^{(m)} \mathbf{H}^{(m)T} \mathbf{H}^{(m)} \, dV^{(m)}}_{= \mathbf{M}^{(m)}}$$

$$\mathbf{R}_B = \sum_m \int_{V^{(m)}} \mathbf{H}^{(m)T} [\mathbf{f}^{B(m)} - \rho^{(m)} \mathbf{H}^{(m)} \ddot{\mathbf{U}} - \kappa^{(m)} \mathbf{H}^{(m)} \dot{\mathbf{U}}] dV^{(m)}$$

$$\mathbf{M} \ddot{\mathbf{U}} + \mathbf{C} \dot{\mathbf{U}} + \mathbf{K} \mathbf{U} = \mathbf{R}$$

$$\mathbf{C} = \sum_m \underbrace{\int_{V^{(m)}} \kappa^{(m)} \mathbf{H}^{(m)T} \mathbf{H}^{(m)} dV^{(m)}}_{\doteq \mathbf{C}^{(m)}}$$

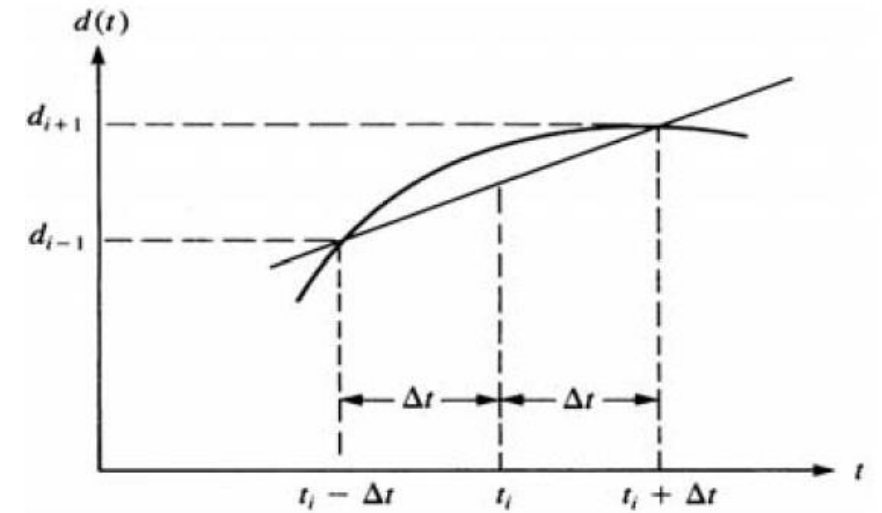
## Central Difference Method

The central difference method is based on finite difference expressions for the derivatives in the equation of motion. For example, consider the velocity and the acceleration at time  $t$ .

$$\{\dot{d}_i\} = \frac{\{d_{i+1}\} - \{d_{i-1}\}}{2(\Delta t)}$$

The acceleration can be expressed in terms of the displacements (using a Taylor series expansion) as:

$$\{\ddot{d}_i\} = \frac{\{d_{i+1}\} - 2\{d_i\} + \{d_{i-1}\}}{(\Delta t)^2}$$



The acceleration can be expressed as:

$$\{\ddot{d}_i\} = [M]^{-1} (\{F_i\} - [K]\{d_i\})$$

$$[M]\{d_{i+1}\} = 2[M]\{d_i\} - [M]\{d_{i-1}\} + (\{F_i\} - [K]\{d_i\})(\Delta t)^2$$

$$[M]\{d_{i+1}\} = (\Delta t)^2 \{F_i\} + [2[M] - (\Delta t)^2 [K]]\{d_i\} - [M]\{d_{i-1}\}$$

To start the computation to determine  $\{d_{i+1}\}$ ,  $\{\dot{d}_{i+1}\}$ , and  $\{\ddot{d}_{i+1}\}$  we need the displacement at time step  $i-1$ .

Using the central difference equations for the velocity and acceleration and solving for  $\{d_{i-1}\}$ :

$$\{d_{i-1}\} = \{d_i\} - (\Delta t)\{\dot{d}_i\} + \{\ddot{d}_i\} \frac{(\Delta t)^2}{2}$$



## Procedure for solution:

1. Given:  $\{d_0\}$ ,  $\{\dot{d}_0\}$ , and  $\{F_i(t)\}$

2. If the acceleration is not given, solve for  $\{\ddot{d}_0\}$

$$\{\ddot{d}_0\} = [M]^{-1} (\{F_0\} - [K]\{d_0\})$$

3. Solve for  $\{d_{-1}\}$  at  $t = -\Delta t$

$$\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \{\ddot{d}_0\} \frac{(\Delta t)^2}{2}$$

4. Solve for  $\{d_1\}$  at  $t = \Delta t$  using the value of  $\{d_{-1}\}$  from Step 3

$$[M]\{d_{i+1}\} = (\Delta t)^2 \{F_i\} + [2[M] - (\Delta t)^2 [K]]\{d_i\} - [M]\{d_{i-1}\}$$

$$\{d_1\} = [M]^{-1} \left\{ (\Delta t)^2 \{F_0\} + [2[M] - (\Delta t)^2 [K]]\{d_0\} - [M]\{d_{-1}\} \right\}$$

5. With  $\{d_0\}$  given and  $\{d_1\}$  determined in Step 4 solve for  $\{d_2\}$

$$\{d_2\} = [\mathbf{M}]^{-1} \left\{ (\Delta t)^2 \{\mathbf{F}_1\} + \left[ 2[\mathbf{M}] - (\Delta t)^2 [\mathbf{K}] \right] \{d_1\} - [\mathbf{M}] \{d_0\} \right\}$$

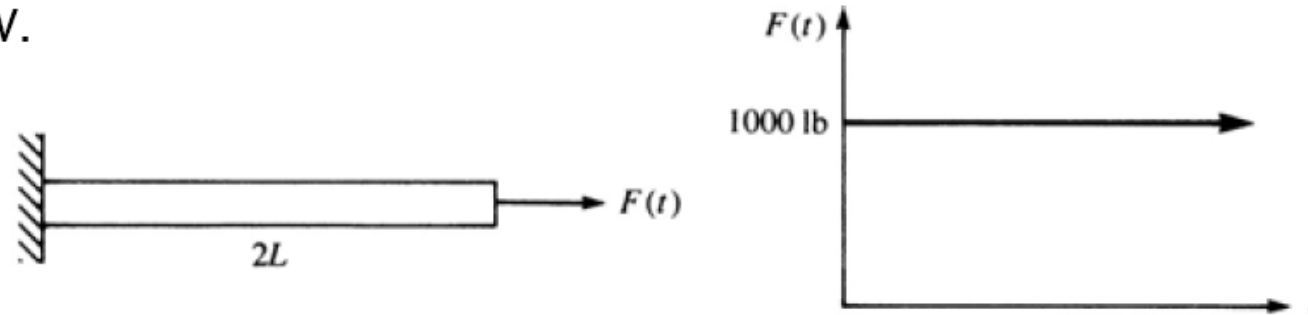
6. Solve for  $\{\ddot{d}_1\}$  :  $\{\ddot{d}_1\} = [\mathbf{M}]^{-1} (\{\mathbf{F}_1\} - [\mathbf{K}] \{d_1\})$

7. Solve for  $\{\dot{d}_1\}$  :  $\{\dot{d}_1\} = \frac{\{d_2\} - \{d_0\}}{2(\Delta t)}$

8. Repeat Steps 5, 6, and 7 to obtain the displacement, acceleration, and velocity for other time steps.

## Time-Dependent One-Dimensional Bar - Example

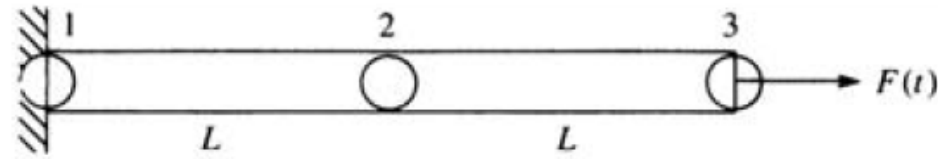
Consider the one-dimensional bar system shown in the figure below.



Assume the boundary condition  $\{d_{1x}\} = 0$  and the initial conditions  $\{d_0\} = 0$  and  $\{\dot{d}_0\} = 0$

Let  $\rho = 0.00073 \text{ lb-s}^2/\text{in}^4$ ,  $A = 1 \text{ in}^2$ ,  $E = 30 \times 10^6 \text{ psi}$ , and  $L = 100 \text{ in}$ .

The bar will be discretized into two elements as shown below.



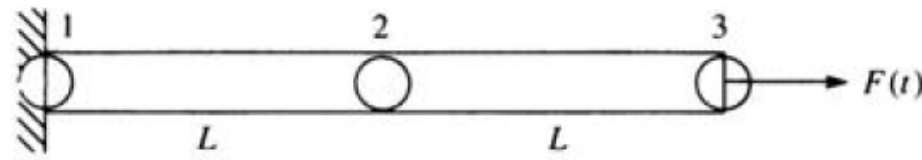
The elemental stiffness matrices are:

$$[k_{(1)}] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_{(2)}] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The global stiffness matrix is:

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$



The lumped-mass matrices are:

$$[m_{(1)}] = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[m_{(2)}] = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The global lumped-mass matrix is:

$$[M] = \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$


---

Substitute the global stiffness and mass matrices into the global dynamic equations gives:

$$\frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \frac{\rho AL}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ F_3(t) \end{Bmatrix}$$

where  $R_1$  denotes the unknown reaction at node 1.

For this example, we will use the central difference method, because it is easier to apply, for the numerical time integration.

It has been mathematically shown that the time step  $\Delta t$  must be less than or equal to two divided by the highest natural frequency.

$$\Delta t \leq \frac{2}{\omega_{max}}$$

For practical results, we should use a time step defined by:

$$\Delta t \leq \frac{3}{4} \left( \frac{2}{\omega_{max}} \right)$$

- An impact problem has high frequencies in the forcing signal which effect the response. So by keeping  $\Delta t$  small we are also ensuring more of these frequencies are participating.



An alternative guide (used only for a bar) for choosing the approximate time step is:

$$\Delta t = \frac{L}{c_x}$$

where  $L$  is the element length, and  $c_x = \sqrt{E_x / \rho}$  is the **longitudinal wave velocity**.

Evaluating the time step estimates gives:

$$\Delta t = \frac{3}{4} \left( \frac{2}{\omega_{max}} \right) = \frac{1.5}{3.76 \times 10^3} = 0.40 \times 10^{-3} \text{ s}$$

$$\Delta t = \frac{L}{c_x} = \frac{100}{\sqrt{30 \times 10^6 / 0.00073}} = 0.48 \times 10^{-3} \text{ s}$$

Applying the boundary conditions  $u_1 = 0$  and  $\ddot{u}_1 = 0$  and simplifying gives:

$$\{\ddot{d}_0\} = \begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_{t=0} = \frac{2000}{\rho AL} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 27,400 \end{Bmatrix} \text{in/s}^2$$

3. Solve for  $d_{-1}$  at  $t = -\Delta t$

$$\{d_{-1}\} = \{d_0\} - (\Delta t)\{\dot{d}_0\} + \{\ddot{d}_0\} \frac{(\Delta t)^2}{2}$$

$$\begin{aligned} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_{-1} &= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - (0.25 \times 10^{-3}) \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \frac{(0.25 \times 10^{-3})^2}{2} \begin{Bmatrix} 0 \\ 27,400 \end{Bmatrix} \\ &= \begin{Bmatrix} 0 \\ 0.856 \times 10^{-3} \end{Bmatrix} \text{in} \end{aligned}$$

4. Solve for  $d_1$  at  $t = \Delta t$  using the value of  $d_{-1}$  from Step 3

$$\{d_1\} = [\mathbf{M}]^{-1} \left\{ (\Delta t)^2 \{F_0\} + \left[ 2[\mathbf{M}] - (\Delta t)^2 [\mathbf{K}] \right] \{d_0\} - [\mathbf{M}] \{d_{-1}\} \right\}$$

$$\begin{aligned} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_1 &= \frac{2}{0.073} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \left\{ (0.25 \times 10^{-3})^2 \begin{Bmatrix} 0 \\ 1,000 \end{Bmatrix} + \left[ \frac{2(0.073)}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right. \right. \\ &\quad \left. \left. - (0.25 \times 10^{-3})^2 (30 \times 10^4) \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right] \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \frac{0.073}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.856 \times 10^{-3} \end{Bmatrix} \right\} \end{aligned}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_1 = \frac{2}{0.073} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \left[ \begin{Bmatrix} 0 \\ 0.0625 \times 10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0.0312 \times 10^{-3} \end{Bmatrix} \right]$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0.858 \times 10^{-3} \end{Bmatrix} \text{ in}$$

Repeating Step 6. Solve for  $\{\ddot{d}_2\}$

$$\{\ddot{d}_2\} = [\mathbf{M}]^{-1} (\{\mathbf{F}_2\} - [\mathbf{K}]\{d_2\})$$

$$\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_2 = \frac{2}{0.073} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1000 \end{Bmatrix} - (30 \times 10^4) \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.221 \times 10^{-3} \\ 2.99 \times 10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}_1 = \begin{Bmatrix} 10,500 \\ 4,600 \end{Bmatrix} \text{ in/s}^2$$

Repeat Step 7: Solve for  $\{\dot{d}_2\}$

$$\{\dot{d}_2\} = \frac{\{d_3\} - \{d_1\}}{2(\Delta t)}$$

$$\begin{Bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix}_2 = \frac{\begin{bmatrix} \begin{Bmatrix} 1.096 \times 10^{-3} \\ 5.397 \times 10^{-3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0.858 \times 10^{-3} \end{Bmatrix} \end{bmatrix}}{2(0.25 \times 10^{-3})} = \begin{Bmatrix} 2.192 \\ 9.078 \end{Bmatrix} in/s$$

## Newmark's Method

Newmark's equations are given as:

$$\{\dot{d}_{i+1}\} = \{\dot{d}_i\} + (\Delta t) \left[ (1-\gamma)\{\ddot{d}_i\} + \gamma\{\ddot{d}_{i+1}\} \right]$$

$$\{d_{i+1}\} = \{d_i\} + (\Delta t)\{\dot{d}_i\} + (\Delta t)^2 \left[ \left(\frac{1}{2} - \beta\right)\{\ddot{d}_i\} + \beta\{\ddot{d}_{i+1}\} \right]$$

where  $\beta$  and  $\gamma$  are parameters.

The parameter  $\beta$  is typically between 0 and  $\frac{1}{4}$ , and  $\gamma$  is often taken to be  $\frac{1}{2}$ .

For example, if  $\beta = 0$  and  $\gamma = \frac{1}{2}$  the above equation reduce to the central difference method.

To find  $\{d_{i+1}\}$  first multiply the above equation by the mass matrix  $[M]$  and substitute the result into this the expression for acceleration. Recall the acceleration is:

$$\{\ddot{d}_i\} = [M]^{-1} (\{F_i\} - [K]\{d_i\})$$

The expression  $[M]\{d_{i+1}\}$  is:

$$\begin{aligned} [M]\{d_{i+1}\} &= [M]\{d_i\} + (\Delta t)[M]\{\dot{d}_i\} + (\Delta t)^2 [M]\left(\frac{1}{2} - \beta\right)\{\ddot{d}_i\} \\ &\quad + \beta(\Delta t)^2 [\{F_{i+1}\} - [K]\{d_{i+1}\}] \end{aligned}$$

Combining terms gives:

$$\begin{aligned} ([M] + \beta(\Delta t)^2 [K])\{d_{i+1}\} &= \beta(\Delta t)^2 \{F_{i+1}\} + [M]\{d_i\} \\ &\quad + (\Delta t)[M]\{\dot{d}_i\} + (\Delta t)^2 [M]\left(\frac{1}{2} - \beta\right)\{\ddot{d}_i\} \end{aligned}$$

---

Dividing the above equation by  $\beta(\Delta t)^2$  gives:  $[\mathbf{K}']\{d_{i+1}\} = \{\mathbf{F}'_{i+1}\}$

where:

$$[\mathbf{K}'] = [\mathbf{K}] + \frac{1}{\beta(\Delta t)^2}[\mathbf{M}]$$

$$\{\mathbf{F}'_{i+1}\} = \{\mathbf{F}_{i+1}\} + \frac{[\mathbf{M}]}{\beta(\Delta t)^2} \left[ \{d_i\} + (\Delta t)\{\dot{d}_i\} + \left(\frac{1}{2} - \beta\right)(\Delta t)^2 \{\ddot{d}_i\} \right]$$

The advantages of using Newmark's method over the central difference method are that Newmark's method can be made unconditionally stable (if  $\beta = 1/4$  and  $\gamma = 1/2$ ) and that larger time steps can be used with better results.



## Newmark's Method

### Procedure for solution:

1. Given:  $\{d_0\}$ ,  $\{\dot{d}_0\}$ , and  $\{F_i(t)\}$
2. If the acceleration is not given, solve for  $\{\ddot{d}_0\}$

$$\{\ddot{d}_0\} = [M]^{-1} (\{F_0\} - [K]\{d_0\})$$

3. Solve for  $\{d_1\}$  at  $t = 0$

$$[K']\{d_1\} = \{F'_1\}$$

4. Solve for  $\{\ddot{d}_1\}$  (original Newmark equation for  $\{d_{i+1}\}$  rewritten for  $\{\ddot{d}_{i+1}\}$ ):

$$\{\ddot{d}_1\} = \frac{1}{\beta(\Delta t)^2} \left[ \{d_1\} - \{d_0\} - (\Delta t)\{\dot{d}_0\} - (\Delta t)^2 \left( \frac{1}{2} - \beta \right) \{\ddot{d}_0\} \right]$$

5. Solve for  $\{\dot{d}_1\}$

$$\{\dot{d}_1\} = \{\dot{d}_0\} + (\Delta t) \left[ (1 - \gamma) \{\ddot{d}_0\} + \gamma \{\ddot{d}_1\} \right]$$

6. Repeat Steps 3, 4, and 5 to obtain the displacement, acceleration, and velocity for the next time step.

Frequencies and mode shapes of a cantilever beam

$$\mathbf{S} = \frac{2EI}{L^3} \begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}$$

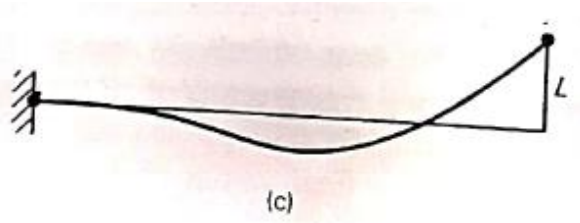
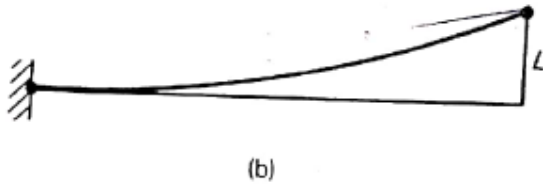
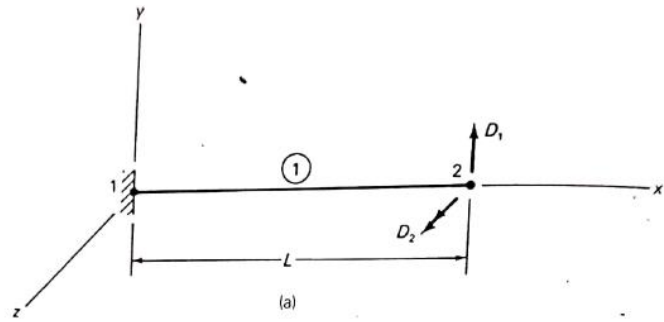
$$\mathbf{M}_t = \frac{\rho AL}{210} \begin{bmatrix} 78 & -11L \\ -11L & 2L^2 \end{bmatrix}$$

Letting  $s = 2EI/L^3$  and  $m = \rho AL/210$

$$\begin{bmatrix} 6(s - 13m\omega_i^2) & -L(3s - 11m\omega_i^2) \\ -L(3s - 11m\omega_i^2) & 2L^2(s - m\omega_i^2) \end{bmatrix} \begin{bmatrix} \Phi_{1i} \\ \Phi_{2i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$35m^2\omega_i^4 - 102ms\omega_i^2 + 3s^2 = 0$$

$$\Phi_1 = \begin{bmatrix} L \\ 1.378 \end{bmatrix} \quad \Phi_2 = \begin{bmatrix} L \\ 7.622 \end{bmatrix}$$

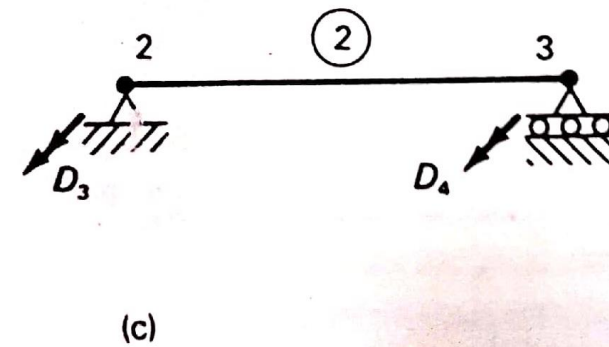
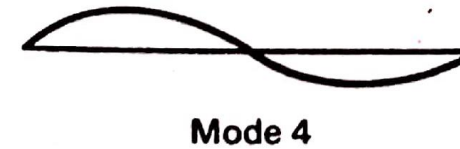
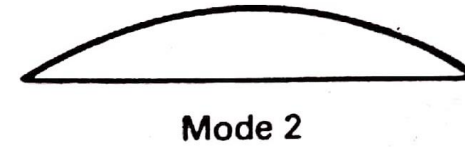
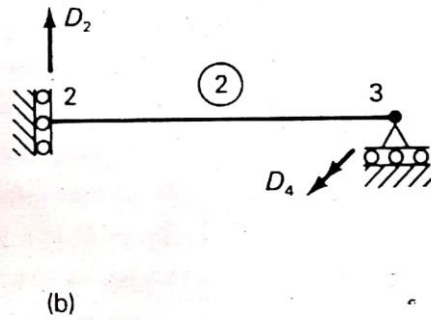
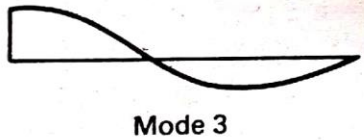
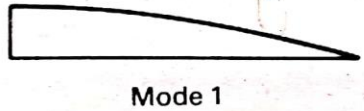
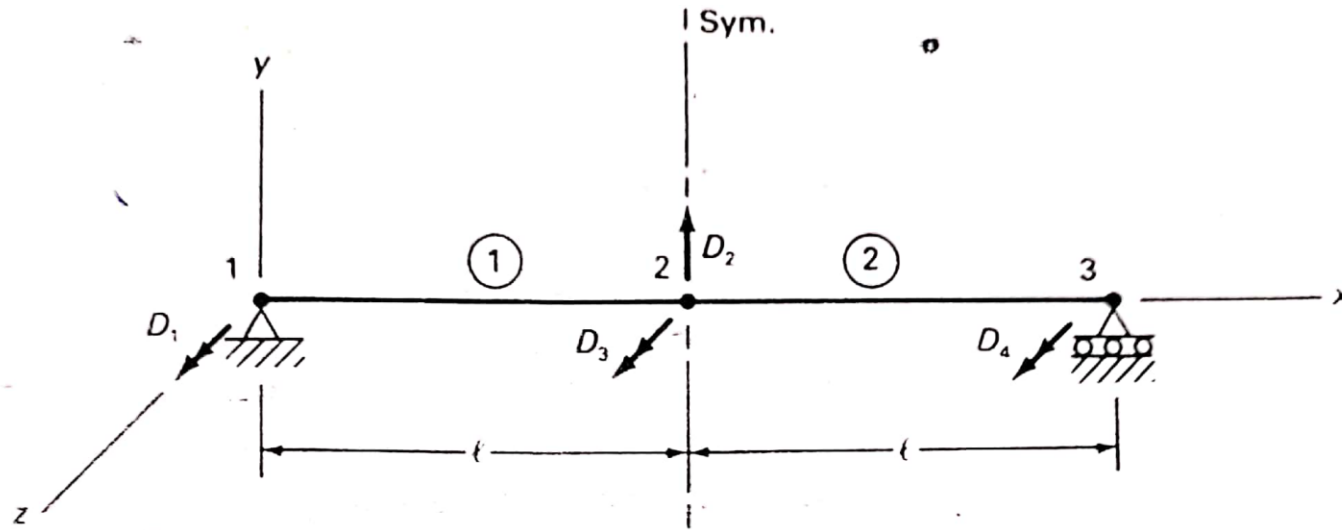


$$\omega_1^2 = 0.02971 \frac{s}{m} \quad \omega_2^2 = 2.885 \frac{s}{m}$$

$$\omega_1 = \frac{3.533}{L^2} \sqrt{\frac{EI}{\rho A}} \quad \omega_2 = \frac{34.81}{L^2} \sqrt{\frac{EI}{\rho A}}$$

$$\Phi_1 = \begin{bmatrix} L \\ 1.378 \end{bmatrix} \quad \Phi_2 = \begin{bmatrix} L \\ 7.622 \end{bmatrix}$$

# Symmetric and Antisymmetric modes



symmetric modes

antisymmetric modes.

$$\mathbf{S} = \frac{2EI}{\ell^3} \begin{bmatrix} 6 & 3\ell \\ 3\ell & 2\ell^2 \end{bmatrix} \quad \mathbf{M}_r = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & -13\ell \\ -13\ell & 4\ell^2 \end{bmatrix}$$

$$\mathbf{H}_i = \mathbf{S} - \omega_i^2 \mathbf{M}_r = \begin{bmatrix} 6(s - 26m\omega_i^2) & \ell(3s + 13m\omega_i^2) \\ \ell(3s + 13m\omega_i^2) & 2\ell^2(s - 2m\omega_i^2) \end{bmatrix}$$

$$s = 2EI/\ell^3 \text{ and } m = \rho A \ell/420$$

$$455m^2\omega_i^4 - 414ms\omega_i^2 + 3s^2 = 0$$

$$\omega_1^2 = 0.007305 \frac{s}{m} \quad \omega_3^2 = 0.9026 \frac{s}{m} \quad (e)$$

Substituting the known values of  $s$  and  $m$  and taking square roots, we find that

$$\omega_1 = \frac{9.909}{L^2} \sqrt{\frac{EI}{\rho A}} \quad \omega_3 = \frac{110.1}{L^2} \sqrt{\frac{EI}{\rho A}} \quad (f)$$

where  $L = 2\ell$ . When these formulas for the angular frequencies are compared with exact values [9], the errors are found to be  $e_1 = +0.40\%$  and  $e_3 = +24\%$ .

We obtain mode shapes corresponding to  $\omega_1$  and  $\omega_3$  using the first column of the adjoint matrix  $\mathbf{H}_i^a$ , as follows:

$$\mathbf{H}_{1i}^a = \begin{bmatrix} 2\ell^2(s - 2m\omega_i^2) \\ -\ell(3s + 13m\omega_i^2) \end{bmatrix} \quad (g)$$

Substitution of  $\omega_1^2$  and  $\omega_3^2$  from Eqs. (e) into this column produces

$$\Phi_1 = \begin{bmatrix} \ell \\ -1.570 \end{bmatrix} \quad \Phi_3 = \begin{bmatrix} \ell \\ 9.149 \end{bmatrix} \quad (h)$$

These mode shapes appear in the left-hand portion of Fig. 3.16(b). Of course, each of them represents half of a symmetric mode shape for the whole beam.

Considering now the antisymmetric case in Fig. 3.16(c), we form the stiffness matrix for the free displacements  $D_3$  and  $D_4$  as

$$\mathbf{S} = \frac{2EI}{\ell} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (i)$$

and the consistent mass matrix is

$$\mathbf{M}_i = \frac{\rho A \ell^3}{420} \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \quad (j)$$

Then the characteristic matrix becomes

$$\mathbf{H}_i = \begin{bmatrix} 2(s - 2m\omega_i^2) & s + 3m\omega_i^2 \\ s + 3m\omega_i^2 & 2(s - 2m\omega_i^2) \end{bmatrix} \quad (k)$$

where  $s$  and  $m$  are the same as before. Expanding the determinant of  $\mathbf{H}_i$  and setting it equal to zero gives the characteristic equation

$$7m^2\omega_i^4 - 22ms\omega_i^2 + 3s^2 = 0 \quad (l)$$

from which the roots are

$$\omega_2^2 = \frac{1}{7} \frac{s}{m} \quad \omega_4^2 = 3 \frac{s}{m} \quad (m)$$

Proceeding as before, we find the angular frequencies to be

$$\omega_2 = \frac{43.82}{L^2} \sqrt{\frac{EI}{\rho A}} \quad \omega_4 = \frac{200.8}{L^2} \sqrt{\frac{EI}{\rho A}} \quad (n)$$

for which the errors are  $+11\%$  and  $+27\%$ .

Mode shapes are given by the first column of  $\mathbf{H}_i^a$ , which is

$$\mathbf{H}_{1i}^a = \begin{bmatrix} 2(s - 2m\omega_i^2) \\ -(s + 3m\omega_i^2) \end{bmatrix} \quad (\text{o})$$

Substituting  $\omega_2^2$  and  $\omega_4^2$  from Eqs. (m) into this vector yields

$$\Phi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \Phi_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\text{p})$$

These mode shapes are displayed in the left-hand part of Fig. 3.16(c), where each of them depicts half of an antisymmetric mode for the whole beam.



# Abaqus

$$\mathbf{M}\ddot{\mathbf{u}} = \mathbf{P} - \mathbf{I}.$$

$$\ddot{\mathbf{u}}|_{(t)} = (\mathbf{M})^{-1} \cdot (\mathbf{P} - \mathbf{I})|_{(t)}$$

$$\dot{\mathbf{u}}|_{(t+\frac{\Delta t}{2})} = \dot{\mathbf{u}}|_{(t-\frac{\Delta t}{2})} + \frac{(\Delta t|_{(t+\Delta t)} + \Delta t|_{(t)})}{2} \ddot{\mathbf{u}}|_{(t)}.$$

$$\mathbf{u}|_{(t+\Delta t)} = \mathbf{u}|_{(t)} + \Delta t|_{(t+\Delta t)} \dot{\mathbf{u}}|_{(t+\frac{\Delta t}{2})}.$$

1. Nodal calculations.

a. Dynamic equilibrium.

$$\ddot{\mathbf{u}}_{(t)} = \mathbf{M}^{-1} (\mathbf{P}_{(t)} - \mathbf{I}_{(t)})$$

b. Integrate explicitly through time.

$$\dot{\mathbf{u}}_{(t+\frac{\Delta t}{2})} = \dot{\mathbf{u}}_{(t-\frac{\Delta t}{2})} + \frac{(\Delta t_{(t+\Delta t)} + \Delta t_{(t)})}{2} \ddot{\mathbf{u}}_t$$

$$\mathbf{u}_{(t+\Delta t)} = \mathbf{u}_{(t)} + \Delta t_{(t+\Delta t)} \dot{\mathbf{u}}_{(t+\frac{\Delta t}{2})}$$