

Figure 5.3. Sixteen-element model

5.6.3 Mixed Formulation for Nearly Incompressible Solids

An effective formulation for analysis of nearly incompressible solids is based on assuming displacements and pressure independently. The formulation is known as *u/p formulation*, with *u* standing for displacement and *p* for pressure. The constitutive equations are separated into deviatoric and volumetric parts. The assumed displacements satisfy the strain-displacement and deviatoric constitutive equations. The pressure constitutive equation is treated as an independent equation. Thus, a weak form is constructed from the following equations:

$$\frac{\partial \sigma_x^d}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial p}{\partial x} + b_x = 0$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y^d}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial p}{\partial y} + b_y = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z^d}{\partial z} + \frac{\partial p}{\partial z} + b_z = 0$$

$$\epsilon_v - \frac{p}{\kappa} = 0$$

Denoting the weighting functions by \bar{u} , \bar{v} , \bar{w} , and \bar{p} , multiplying each equation by its weighting function, integrating over the volume, and adding all four terms, the total weighted residual is

$$\iiint_V \left(\frac{\partial \sigma_x^d}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \frac{\partial p}{\partial x} + b_x \right) \bar{u} + \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y^d}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial p}{\partial y} + b_y \right) \bar{v} + \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z^d}{\partial z} + \frac{\partial p}{\partial z} + b_z \right) \bar{w} + \left(\epsilon_v - \frac{p}{\kappa} \right) \bar{p} dV = 0$$



Using the Green-Gauss theorem on each of the stress derivative terms, we get

$$\begin{aligned} & \iint_S (\sigma_x^d n_x + p n_x + \tau_{xy} n_y + \tau_{xz} n_z) \bar{u} + (\tau_{yx} n_x + \sigma_y^d n_y + p n_y + \tau_{yz} n_z) \bar{v} \\ & + (\tau_{zx} n_x + \tau_{zy} n_y + \sigma_z^d n_z + p n_z) \bar{w} dS - \iiint_V \left(\sigma_x^d \frac{\partial \bar{u}}{\partial x} + \tau_{xy} \frac{\partial \bar{u}}{\partial y} + \tau_{xz} \frac{\partial \bar{u}}{\partial z} + p \frac{\partial \bar{u}}{\partial x} \right) \\ & + \left(\tau_{yx} \frac{\partial \bar{v}}{\partial x} + \sigma_y^d \frac{\partial \bar{v}}{\partial y} + \tau_{yz} \frac{\partial \bar{v}}{\partial z} + p \frac{\partial \bar{v}}{\partial y} \right) + \left(\tau_{zx} \frac{\partial \bar{w}}{\partial x} + \tau_{zy} \frac{\partial \bar{w}}{\partial y} + \sigma_z^d \frac{\partial \bar{w}}{\partial z} + p \frac{\partial \bar{w}}{\partial z} \right) - \left(\epsilon_v - \frac{p}{\kappa} \right) \bar{p} dV \\ & + \iiint_V (b_x \bar{u} + b_y \bar{v} + b_z \bar{w}) dV = 0 \end{aligned}$$

On the surface the applied forces are

$$\sigma_x^d n_x + p n_x + \tau_{xy} n_y + \tau_{xz} n_z = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \equiv q_x$$

and so on. Substituting these and rearranging terms, we get the weak form:

$$\begin{aligned} & \iiint_V \sigma_x^d \frac{\partial \bar{u}}{\partial x} + \sigma_y^d \frac{\partial \bar{v}}{\partial y} + \sigma_z^d \frac{\partial \bar{w}}{\partial z} + \tau_{xy} \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) + \tau_{yz} \left(\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} \right) + \tau_{zx} \left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right) \\ & + p \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \right) - \left(\epsilon_v - \frac{p}{\kappa} \right) \bar{p} dV \\ & = \iint_S (q_x \bar{u} + q_y \bar{v} + q_z \bar{w}) dS + \iiint_V (b_x \bar{u} + b_y \bar{v} + b_z \bar{w}) dV \end{aligned}$$

If we interpret the weighting functions as virtual displacements, their derivatives are virtual strains:

$$\begin{aligned} \frac{\partial \bar{u}}{\partial x} &= \bar{\epsilon}_x; & \frac{\partial \bar{v}}{\partial y} &= \bar{\epsilon}_y; & \frac{\partial \bar{w}}{\partial z} &= \bar{\epsilon}_z \\ \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} &= \bar{\gamma}_{xy}; & \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} &= \bar{\gamma}_{yz}; & \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} &= \bar{\gamma}_{zx} \\ \bar{\epsilon}_v &= \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} \end{aligned}$$

Substituting these, the weak form is

$$\begin{aligned} & \iiint_V \sigma_x^d \bar{\epsilon}_x + \sigma_y^d \bar{\epsilon}_y + \sigma_z^d \bar{\epsilon}_z + \tau_{xy} \bar{\gamma}_{xy} + \tau_{yz} \bar{\gamma}_{yz} + \tau_{zx} \bar{\gamma}_{zx} + p \bar{\epsilon}_v - \left(\epsilon_v - \frac{p}{\kappa} \right) \bar{p} dV \\ & = \iint_S (q_x \bar{u} + q_y \bar{v} + q_z \bar{w}) dS + \iiint_V (b_x \bar{u} + b_y \bar{v} + b_z \bar{w}) dV \end{aligned}$$



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By A'Awasthi

Using usual vector notation for displacement, stresses, and so on, we have

$$\bar{\epsilon} = \left(\frac{\partial \bar{u}}{\partial x}, \frac{\partial \bar{v}}{\partial y}, \frac{\partial \bar{w}}{\partial z}, \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}, \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y}, \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \right)^T$$

$$\sigma_d = (\sigma_x^d, \sigma_y^d, \sigma_z^d, \tau_{xy}, \tau_{yz}, \tau_{zx})^T$$

$$q^T = (q_x, q_y, q_z); \quad \bar{u}^T = (\bar{u}, \bar{v}, \bar{w}); \quad b^T = (b_x, b_y, b_z)$$

The weak form therefore is

$$\iiint_V \bar{\epsilon}^T \sigma_d + \bar{\epsilon}_v p - \bar{p} \left(\epsilon_v - \frac{p}{\kappa} \right) dV = \iint_S \bar{u}^T q dS + \iiint_V \bar{u}^T b dV$$

Introducing the virtual deviatoric strain vector,

$$\bar{\epsilon}_d = \bar{\epsilon} - \frac{1}{3} m \bar{\epsilon}_v \implies \bar{\epsilon} = \bar{\epsilon}_d + \frac{1}{3} m \bar{\epsilon}_v$$

the weak form becomes

$$\iiint_V \bar{\epsilon}_d^T \sigma_d + \frac{1}{3} \bar{\epsilon}_v m^T \sigma_d + \bar{\epsilon}_v p - \bar{p} \left(\epsilon_v - \frac{p}{\kappa} \right) dV = \iint_S \bar{u}^T q dS + \iiint_V \bar{u}^T b dV$$

By direct multiplication it can easily be verified that

$$m^T \sigma_d = 0$$

Thus, the final weak form is

$$\iiint_V \bar{\epsilon}_d^T \sigma_d + \bar{\epsilon}_v p - \bar{p} \left(\epsilon_v - \frac{p}{\kappa} \right) dV = \iint_S \bar{u}^T q dS + \iiint_V \bar{u}^T b dV$$

5.6.4 Finite Element Equations

For developing the finite element equations, we need three displacement interpolations and one pressure interpolation:

$$u \equiv \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} N_1 & 0 & 0 & N_2 & 0 & \dots \\ 0 & N_1 & 0 & 0 & N_2 & \dots \\ 0 & 0 & N_1 & 0 & 0 & \dots \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ \vdots \end{pmatrix} \equiv N^T d$$

$$p = P^T \beta$$

where $u_1, v_1, w_1, u_2, \dots$ are the n nodal degrees of freedom $N_i(x, y, z)$ are suitable interpolation functions, P^T is a suitable $1 \times m$ matrix of pressure interpolation functions, and β is



vector of parameters. Thus, the complete set of assumed solutions is

$$\begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} 0 & N^T \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \beta \\ d \end{pmatrix}$$

The strains from the assumed displacements are written

$$\boldsymbol{\epsilon} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & \frac{\partial N_2}{\partial x} & 0 & 0 & \dots \\ 0 & \frac{\partial N_1}{\partial y} & 0 & 0 & \frac{\partial N_2}{\partial y} & 0 & \dots \\ 0 & 0 & \frac{\partial N_1}{\partial z} & 0 & 0 & \frac{\partial N_2}{\partial z} & \dots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & 0 & \dots \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial y} & \dots \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_2}{\partial x} & \dots \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ \vdots \end{pmatrix} \equiv \mathbf{B}^T \mathbf{d}$$

To split this into volumetric and deviatoric strains, we note that a direct multiplication using the vector \mathbf{m} the volumetric strain can be obtained from the vector of total strains as follows:

$$\epsilon_v = \mathbf{m}^T \boldsymbol{\epsilon} = \mathbf{m}^T \mathbf{B}^T \mathbf{d} \equiv \mathbf{B}_v^T \mathbf{d} \Rightarrow \mathbf{B}_v^T = \mathbf{m}^T \mathbf{B}^T$$

The deviatoric strains are expressed as

$$\begin{aligned} \boldsymbol{\epsilon}_d &= \boldsymbol{\epsilon} - \frac{1}{3} \mathbf{m} \epsilon_v = \boldsymbol{\epsilon} - \frac{1}{3} \mathbf{m} \mathbf{m}^T \boldsymbol{\epsilon} = (\mathbf{I} - \frac{1}{3} \mathbf{m} \mathbf{m}^T) \boldsymbol{\epsilon} = (\mathbf{I} - \frac{1}{3} \mathbf{m} \mathbf{m}^T) \mathbf{B}^T \mathbf{d} \equiv \mathbf{B}_d^T \mathbf{d} \\ &\Rightarrow \mathbf{B}_d^T = (\mathbf{I} - \frac{1}{3} \mathbf{m} \mathbf{m}^T) \mathbf{B}^T \end{aligned}$$

\mathbf{I} = 6×6 identity matrix

The deviatoric stresses can be obtained from the deviatoric strains by using the constitutive equations:

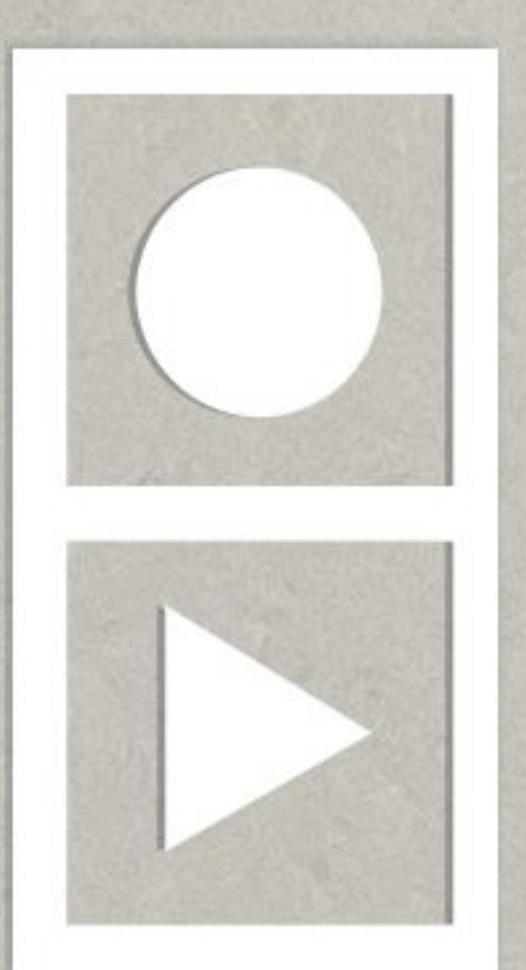
$$\boldsymbol{\sigma}_d = \mathbf{C}_d \boldsymbol{\epsilon}_d = \mathbf{C}_d \mathbf{B}_d^T \mathbf{d}$$

Substituting the assumed solutions in the weak form, we have

$$\iiint_V \bar{\boldsymbol{\epsilon}}_d^T \mathbf{C}_d \mathbf{B}_d^T \mathbf{d} + \bar{\epsilon}_v \mathbf{P}^T \beta - \bar{p} \left(\mathbf{B}_v^T \mathbf{d} - \frac{1}{\kappa} \mathbf{P}^T \beta \right) dV = \iint_S \bar{\mathbf{u}}^T \mathbf{q} dS + \iiint_V \bar{\mathbf{u}}^T \mathbf{b} dV$$

The sets of weighting functions are the columns of the matrix of the assumed solution functions:

$$\begin{pmatrix} \bar{\mathbf{u}} \\ \bar{p} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & N^T \\ P^T & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{\epsilon}_d \\ \bar{\epsilon}_v \\ \bar{p} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \mathbf{B}_d^T \\ 0 & \mathbf{B}_v^T \\ P^T & 0 \end{pmatrix}$$



Substituting the first set of the weighting functions into the weak form, we get

$$\iiint_V -P \left(B_v^T d - \frac{1}{\kappa} P^T \beta \right) dV = 0$$

Rearranging terms, we get

$$-\iiint_V PB_v^T dV d + \iiint_V P \frac{1}{\kappa} P^T dV \beta = 0 \Rightarrow k_a d - k_b \beta = 0$$

where

$$k_a = \iiint_V PB_v^T dV \quad \text{and} \quad k_b = \iiint_V P \frac{1}{\kappa} P^T dV$$

Substituting the second set of the weighting functions into the weak form, we get the equations

$$\iiint_V B_d C_d B_d^T d + B_v P^T \beta dV = \iint_S Nq dS + \iiint_V Nb dV$$

$$k_c d + k_a^T \beta = r_q + r_b \equiv r$$

where

$$k_c = \iiint_V B_d C_d B_d^T dV; \quad k_a^T = \iiint_V B_v P^T dV; \quad r_q = \iint_S Nq dS; \quad r_b = \iiint_V Nb dV$$

Writing the two sets of equations together, we get the following system of element equations:

$$\begin{pmatrix} -k_b & k_a \\ k_a^T & k_c \end{pmatrix} \begin{pmatrix} \beta \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

These equations can be assembled in a usual manner to get global equations. After introducing boundary conditions these equations can be solved for the unknown parameters. If continuity of stress parameters is not to be maintained across element interfaces, we can use static-condensation to eliminate the stress parameters from the element equations before assembly. The first matrix equation gives

$$(-k_b \quad k_a) \begin{pmatrix} \beta \\ d \end{pmatrix} = 0 \Rightarrow \beta = k_b^{-1} k_a d$$



The second matrix equation gives

$$\mathbf{k}_a^T \boldsymbol{\beta} + \mathbf{k}_c \mathbf{d} = \mathbf{r} \Rightarrow (\mathbf{k}_c + \mathbf{k}_a^T \mathbf{k}_b^{-1} \mathbf{k}_a) \mathbf{d} = \mathbf{r} \Rightarrow \mathbf{k} \mathbf{d} = \mathbf{r}$$

where $\mathbf{k} = \mathbf{k}_c + \mathbf{k}_a^T \mathbf{k}_b^{-1} \mathbf{k}_a$ is the effective stiffness matrix in terms of displacement degrees of freedom. These equations are now treated in exactly the same manner as conventional displacement-based elements. After solving for nodal displacements, the stress parameters for each element can be computed from the first equation, if desired. However, for nearly incompressible materials, it is more convenient simply to use the standard displacement-based approach to compute the element quantities. This procedure is used in the numerical examples that are presented in the following sections.

For a fully incompressible material, since $\kappa = \infty$, the matrix \mathbf{k}_b is a zero matrix. Therefore, the element equations are as follows:

$$\begin{pmatrix} 0 & \mathbf{k}_a \\ \mathbf{k}_a^T & \mathbf{k}_c \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{r} \end{pmatrix}$$

In this situation we cannot use the static condensation to eliminate the stress parameters. We must assemble the element equations with both displacement and stress parameters as unknowns. The solution for nodal displacements and stress parameters is obtained directly from the global equations. The solution over each element must be computed by separating the stresses and strains into deviatoric and volumetric parts. The pressure must be computed from the assumed interpolation relationship.

5.6.5 Assumed Pressure Solution

As is true for all mixed formulations, the success of elements for analysis of nearly incompressible solids depends on the assumed pressure solution. Using a too-high-order polynomial for pressure will make the element behave similar to its displacement-based counterpart and hence exhibit locking. If the order of pressure interpolation is too low, the solution may give poor prediction for pressure and, even worse, may cause *spurious pressure modes*. A spurious pressure mode corresponds to the case when there is a nonzero vector of pressure parameters such that

$$\mathbf{k}_b \boldsymbol{\beta} = \mathbf{0}$$

The system of equations is unsolvable when this condition is present. A mathematical criterion known as the *inf-sup condition* (also known as the *Babuska-Brezzi condition*) has been developed to check whether a mixed formulation will result in a system of equations that is solvable. The criterion is not very intuitive. Its derivation is based on technical concepts from functional analysis. Therefore, it is not discussed here. A readable treatment, with some numerical examples, can be found in K. J. Bathe, *Finite Element Procedures* (Upper Saddle River, NJ: Prentice Hall, 1996). For complete mathematical details see Brezzi and Fortin, *Mixed and Hybrid Finite Element Methods* (New York: Springer-Verlag, 1991), and S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 2nd edition (New York: Springer-Verlag, 2002).

5.6.6 Quadrilateral Elements for Planar Problems

The general finite element equations developed in Section 5.6.5 are specialized in this section for plane stress and plane strain problems. Recall that the plane stress and plane strain formulations are based on the following assumptions:

$$\text{Plane stress: } \sigma_z = \tau_{yz} = \tau_{zx} = 0 \implies \epsilon_z = -\frac{\nu(\sigma_x + \sigma_y)}{E} \equiv \frac{\nu(\epsilon_x + \epsilon_y)}{\nu - 1}; \quad \gamma_{yz} = 0; \quad \gamma_{zx} = 0$$

$$\text{Plane strain: } \epsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \implies \sigma_z = \frac{E\nu(\epsilon_x + \epsilon_y)}{(1 + \nu)(1 - 2\nu)}; \quad \tau_{yz} = 0; \quad \tau_{zx} = 0$$

In the usual formulation for plane stress and plane strain elements, ϵ_z and σ_z do enter into the equations. Therefore, the element equations are based on considering only three stress ($\sigma_x, \sigma_y, \tau_{xy}$) and corresponding strain components. However, in the formulation for incompressible materials, the volumetric strain involves all three normal strain components. Since $\epsilon_z \neq 0$ for plane stress, for a unified plane stress/strain element, we must include four strain and corresponding stress components explicitly in the formulation.

Element strain vector for plane stress:

$$\begin{aligned} \boldsymbol{\epsilon} &= \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\nu}{\nu - 1} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots \\ \frac{\nu}{\nu - 1} \frac{\partial N_1}{\partial x} & \frac{\nu}{\nu - 1} \frac{\partial N_1}{\partial y} & \frac{\nu}{\nu - 1} \frac{\partial N_2}{\partial x} & \frac{\nu}{\nu - 1} \frac{\partial N_2}{\partial y} & \dots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{pmatrix} = \mathbf{B}^T \mathbf{d} \end{aligned}$$

Element strain vector for plane strain:

$$\begin{aligned} \boldsymbol{\epsilon} &= \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \dots \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \dots \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{pmatrix} = \mathbf{B}^T \mathbf{d} \end{aligned}$$

where x_i and determinant

Knowing the then be evaluated

where $m^T =$
obtained by using
volume integrals.

Assumed solution over parent element:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} N_1 & 0 & N_2 & 0 & \dots \\ 0 & N_1 & 0 & N_2 & \dots \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{pmatrix} = N^T d$$

$$p = P^T \beta$$

The derivatives of the interpolation functions with respect to x and y are computed using the mapping. The mapping is

$$x = (N_1 \quad N_2 \quad \dots) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \quad \text{and} \quad y = (N_1 \quad N_2 \quad \dots) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix}$$

where x_i and y_i are the nodal coordinates. The Jacobian matrix of the mapping and its determinant are as follows:

$$J = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}; \quad \det J = \left| \frac{\partial x}{\partial s} \frac{\partial y}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial y}{\partial s} \right|$$

The derivatives of the interpolation functions with respect to x and y are computed as follows:

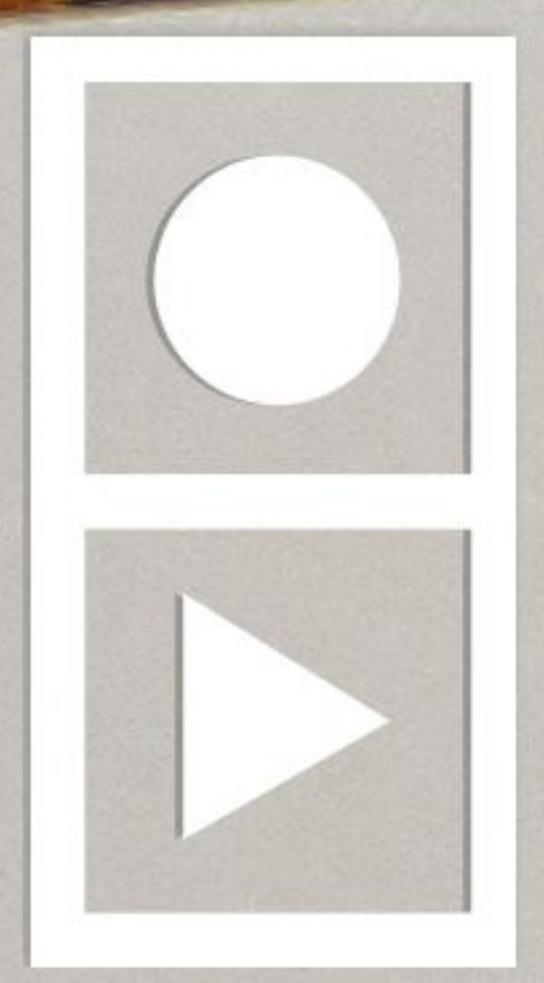
$$\frac{\partial N_i}{\partial x} = \frac{1}{\det J} \left(J_{22} \frac{\partial N_i}{\partial s} - J_{21} \frac{\partial N_i}{\partial t} \right)$$

$$\frac{\partial N_i}{\partial y} = \frac{1}{\det J} \left(-J_{12} \frac{\partial N_i}{\partial s} + J_{11} \frac{\partial N_i}{\partial t} \right)$$

Knowing the B matrix, the deviatoric and volumetric strain-displacement matrices can then be evaluated:

$$B_v^T = m^T B^T \quad \text{and} \quad B_d^T = \left(I - \frac{1}{3} m m^T \right) B^T$$

where $m^T = (1 \ 1 \ 1 \ 0)$ and I is a 4×4 identity matrix. The element equations are obtained by using the numerical integration as usual. With a constant thickness h , the volume integration reduces to area integration and the surface integration to line integrals.



Element stiffness matrix:

$$k_a = h \iint_S PB_v^T dA = h \int_{-1}^1 \int_{-1}^1 PB_v^T \det J ds dt$$

$$= h \sum_{i=1}^m \sum_{j=1}^n w_i w_j P(s_i, t_j) B_v^T(s_i, t_j) \det J(s_i, t_j)$$

$$k_b = h \iint_S P \frac{1}{\kappa} P^T dA = h \int_{-1}^1 \int_{-1}^1 P \frac{1}{\kappa} P^T \det J ds dt$$

$$= h \sum_{i=1}^m \sum_{j=1}^n w_i w_j P(s_i, t_j) \frac{1}{\kappa} P^T(s_i, t_j) \det J(s_i, t_j)$$

$$k_c = h \iint_S B_d C_d B_d^T dA = h \int_{-1}^1 \int_{-1}^1 B_d C_d B_d^T \det J ds dt$$

$$= h \sum_{i=1}^m \sum_{j=1}^n w_i w_j B_d(s_i, t_j) C_d B_d^T(s_i, t_j) \det J(s_i, t_j)$$

$$k = k_c + k_a^T k_b^{-1} k_a$$

$$C_d = \begin{pmatrix} 2G & 0 & 0 & 0 \\ 0 & 2G & 0 & 0 \\ 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & G \end{pmatrix}; \quad G = \frac{E}{2(1+\nu)}; \quad \kappa = \frac{E}{3(1-2\nu)}$$

Equivalent load vector due to body forces:

$$r_b = h \iint_S N \begin{pmatrix} b_x \\ b_y \end{pmatrix} dA = h \int_{-1}^1 \int_{-1}^1 N \begin{pmatrix} b_x \\ b_y \end{pmatrix} \det J ds dt$$

$$= h \sum_{i=1}^m \sum_{j=1}^n w_i w_j N(s_i, t_j) \begin{pmatrix} b_x \\ b_y \end{pmatrix} \det J(s_i, t_j)$$

Equivalent load vector due to distributed loads:

$$r_q = h \int_c N_c \begin{pmatrix} q_x \\ q_y \end{pmatrix} dc = h \int_{-1}^1 N_c \begin{pmatrix} q_x \\ q_y \end{pmatrix} J_c da = h \sum_{i=1}^n w_i N_c(a_i) \begin{pmatrix} q_x \\ q_y \end{pmatrix} J_c(a_i)$$

where the subscript c indicates that for evaluation of boundary integrals the interpolation functions must be written in terms of a coordinate along that side.

$$B_v^T = \begin{pmatrix} -0.0 \\ 0 \\ 0.9 \\ -0.0 \end{pmatrix}$$

$$B_v^T = \{-0.0\}$$

$$k_a = (-0.00)$$

$$k_b = (1.5 \times)$$

$$k_c = \begin{pmatrix} 252 \\ 148 \\ -186 \\ -26 \\ -67 \\ -39 \\ 1 \\ -82 \end{pmatrix}$$

Performing
sulting ma

$$k_a = (-0.019)$$

$$k_b = (6 \times 10^-)$$

$$k_c = \begin{pmatrix} 5419 \\ 2386 \\ -3741 \\ 7090 \\ -27098 \\ -2386 \\ 10319 \\ -7090 \end{pmatrix}$$

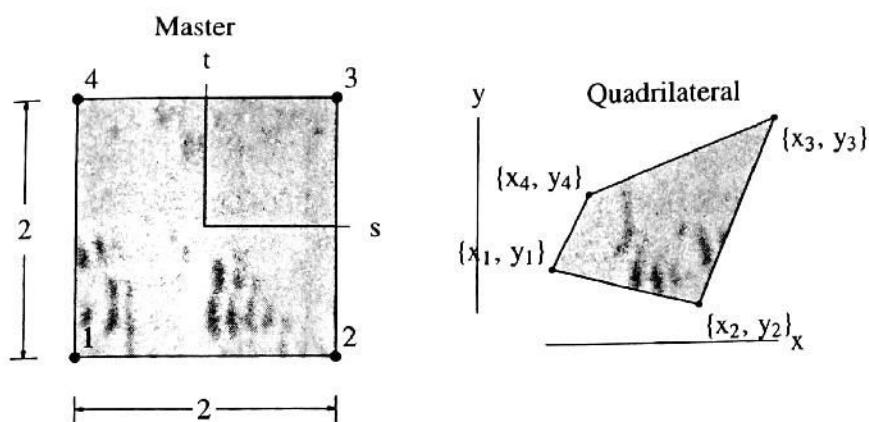


Figure 5.4. Four-node master and actual quadrilateral element

4/1-u/p Quadrilateral Element The simplest *u/p* element is a four-node quadrilateral element as shown in Figure 5.4. The 4/1 terminology is meant to indicate four displacement interpolation functions and one pressure term. The interpolation functions for mapping and the displacements are

$$\begin{pmatrix} \frac{1}{4}(1-s)(1-t) \\ \frac{1}{4}(s+1)(1-t) \\ \frac{1}{4}(s+1)(t+1) \\ \frac{1}{4}(1-s)(t+1) \end{pmatrix}$$

A constant pressure is assumed over the element. Thus, the stress interpolation matrix is a 1×1 matrix with the value 1. That is,

$$P^T = (1)$$

The element is popular because of its simplicity. For most practical problems it gives reasonable results. However, it has been shown that the element does not pass the inf-sup test. Therefore, it must be used with caution. Examples exist in which the results obtained with this element, instead of improving, get worse as the mesh is refined.

Example 5.3 Use only one 4/1 element to analyze the square cantilever plate shown in Figure 5.5. Assume that $L = 10$ in, $E = 10^6$ lb/in 2 , $\nu = 0.49$, $q = 1000$ lb/in 2 , and plate thickness, $h = 0.1$ in.

$$\text{Plane stress } C_d = \begin{pmatrix} 671141 & 0 & 0 & 0 \\ 0 & 671141 & 0 & 0 \\ 0 & 0 & 671141 & 0 \\ 0 & 0 & 0 & 335570 \end{pmatrix}$$

$$\kappa = 1.66667 \times 10^7; \quad G = 335570$$

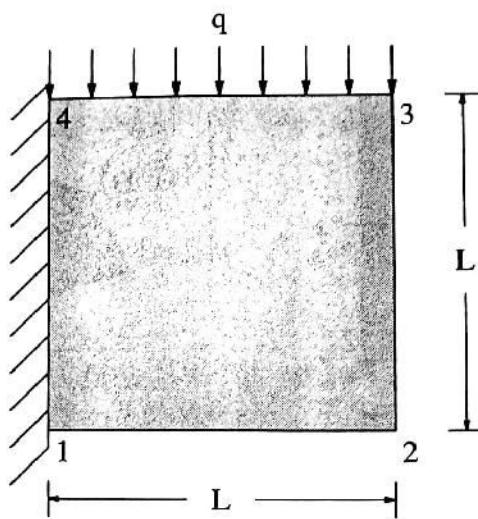


Figure 5.5. Cantilever plate

Interpolation functions and their derivatives:

$$\mathbf{N}^T = \left\{ \frac{1}{4}(s-1)(t-1), -\frac{1}{4}(s+1)(t-1), \frac{1}{4}(s+1)(t+1), -\frac{1}{4}(s-1)(t+1) \right\}$$

$$\frac{\partial \mathbf{N}^T}{\partial s} = \left\{ \frac{t-1}{4}, \frac{1-t}{4}, \frac{t+1}{4}, \frac{1}{4}(-t-1) \right\}$$

$$\frac{\partial \mathbf{N}^T}{\partial t} = \left\{ \frac{s-1}{4}, \frac{1}{4}(-s-1), \frac{s+1}{4}, \frac{1-s}{4} \right\}$$

$$\mathbf{P}^T = \{1\}$$

Mapping to the master element yields

$$x(s, t) = 5s + 5$$

$$y(s, t) = 5t + 5$$

$$\mathbf{J} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \quad \det \mathbf{J} = 25$$

Gauss quadrature points and weights:

	Point	Weight
1	$s \rightarrow -0.57735$ $t \rightarrow -0.57735$	1
2	$s \rightarrow -0.57735$ $t \rightarrow 0.57735$	1
3	$s \rightarrow 0.57735$ $t \rightarrow -0.57735$	1
4	$s \rightarrow 0.57735$ $t \rightarrow 0.57735$	1

Computation of element matrices at $\{-0.57735, -0.57735\}$ with weight = 1:

$$J = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \det J = 25$$

$$N^T = (0.622008 \quad 0.166667 \quad 0.0446582 \quad 0.166667)$$

$$\frac{\partial N^T}{\partial s} = (-0.394338 \quad 0.394338 \quad 0.105662 \quad -0.105662)$$

$$\frac{\partial N^T}{\partial t} = (-0.394338 \quad -0.105662 \quad 0.105662 \quad 0.394338)$$

$$B^T = \begin{pmatrix} -0.0788675 & 0 & 0.0788675 & 0 & 0.0211325 & 0 & -0.0211325 & 0 \\ 0 & -0.0788675 & 0 & -0.0211325 & 0 & 0.0211325 & 0 & 0.0788675 \\ 0.0757747 & 0.0757747 & -0.0757747 & 0.0203038 & -0.0203038 & -0.0203038 & 0.0203038 & -0.0757747 \\ -0.0788675 & -0.0788675 & -0.0211325 & 0.0788675 & 0.0211325 & 0.0211325 & 0.0788675 & -0.0211325 \end{pmatrix}$$

$$B_v^T = \{-0.00309284, -0.00309284, 0.00309284, 0.000828725, 0.000828725, 0.000828725, -0.000828725, 0.00309284\}$$

$$B_d^T = \begin{pmatrix} -0.0778366 & 0.00103095 & 0.0778366 & 0.000276242 & 0.0208562 & -0.000276242 & -0.0208562 \\ -0.00103095 & -0.0778366 & -0.00103095 & -0.0208562 & -0.000276242 & 0.0208562 & 0.000276242 \\ 0.0768056 & 0.0768056 & -0.0768056 & 0.02058 & -0.02058 & -0.02058 & 0.02058 \\ -0.0788675 & -0.0788675 & -0.0211325 & 0.0788675 & 0.0211325 & 0.0211325 & 0.0788675 \end{pmatrix}$$

$$P^T = (1)$$

$$k_a = (-0.00773211 \quad -0.00773211 \quad 0.00773211 \quad -0.00207181 \quad 0.00207181 \quad 0.00207181 \quad -0.00207181 \quad 0.00773211)$$

$$k_b = (1.5 \times 10^{-7})$$

$$E \quad -2\nu$$

$$k_c = \begin{pmatrix} 25283.1 & 14846.7 & -18666.7 & -2638.23 & -6774.59 & -3978.17 & 158.189 & -8230.33 \\ 14846.7 & 25283.1 & -8230.33 & 158.189 & -3978.17 & -6774.59 & -2638.23 & -18666.7 \\ -18666.7 & -8230.33 & 20439.6 & -3978.17 & 5001.73 & 2205.31 & -6774.59 & 10003.2 \\ -2638.23 & 158.189 & -3978.17 & 6658.79 & 706.912 & -42.3867 & 5909.49 & -6774.59 \\ -6774.59 & -3978.17 & 5001.73 & 706.912 & 1815.25 & 1065.95 & -42.3867 & 2205.31 \\ -3978.17 & -6774.59 & 2205.31 & -42.3867 & 1065.95 & 1815.25 & 706.912 & 5001.73 \\ 158.189 & -2638.23 & -6774.59 & 5909.49 & -42.3867 & 706.912 & 6658.79 & -3978.17 \\ -8230.33 & -18666.7 & 10003.2 & -6774.59 & 2205.31 & 5001.73 & -3978.17 & 20439.6 \end{pmatrix}$$

Performing these computations at the remaining three Gauss points and adding the resulting matrices, we get

$$k_a = (-0.0196078 \quad -0.0196078 \quad 0.0196078 \quad -0.0196078 \quad 0.0196078 \quad 0.0196078 \quad -0.0196078 \quad 0.0196078)$$

$$k_b = (6 \times 10^{-7})$$

$$k_c = \begin{pmatrix} 54196.7 & 23869 & -37418.2 & 7090.5 & -27098.4 & -23869 & 10319.8 & -7090.5 \\ 23869 & 54196.7 & -7090.5 & 10319.8 & -23869 & -27098.4 & 7090.5 & -37418.2 \\ -37418.2 & -7090.5 & 54196.7 & -23869 & 10319.8 & 7090.5 & -27098.4 & 23869 \\ 7090.5 & 10319.8 & -23869 & 54196.7 & -7090.5 & -37418.2 & 23869 & -27098.4 \\ -27098.4 & -23869 & 10319.8 & -7090.5 & 54196.7 & 23869 & -37418.2 & 7090.5 \\ -23869 & -27098.4 & 7090.5 & -37418.2 & 23869 & 54196.7 & -7090.5 & 10319.8 \\ 10319.8 & 7090.5 & -27098.4 & 23869 & -37418.2 & -7090.5 & 54196.7 & -23869 \\ -7090.5 & -37418.2 & 23869 & -27098.4 & 7090.5 & 10319.8 & -23869 & 54196.7 \end{pmatrix}$$

$$k = \begin{pmatrix} 54837.5 & 24509.8 & -38059 & 7731.28 & -27739.1 & -24509.8 & 10960.6 & -7731.28 \\ 24509.8 & 54837.5 & -7731.28 & 10960.6 & -24509.8 & 10960.6 & 7731.28 & -38059 \\ -38059 & -7731.28 & 54837.5 & -24509.8 & 54837.5 & -7731.28 & -38059 & 24509.8 \\ 7731.28 & 10960.6 & -24509.8 & 54837.5 & 24509.8 & -38059 & -7731.28 & -27739.1 \\ -27739.1 & -24509.8 & 10960.6 & -7731.28 & 24509.8 & 54837.5 & 10960.6 & 7731.28 \\ -24509.8 & -27739.1 & 7731.28 & -38059 & -38059 & -7731.28 & 54837.5 & -24509.8 \\ 10960.6 & 7731.28 & -27739.1 & 24509.8 & 24509.8 & 10960.6 & -24509.8 & 54837.5 \\ -7731.28 & -38059 & 24509.8 & -27739.1 & 7731.28 & 10960.6 & 54837.5 & -38059 \end{pmatrix}$$

$$r = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$$

Computation of element matrices resulting from NBC on side 3 with $q = \{-1000, 0\}$:

$$N_c^T = \begin{pmatrix} 0 & 0 & \frac{1-a}{2} & \frac{a+1}{2} \end{pmatrix}$$

$$x(a) = 5 - 5a; \quad y(a) = 10;$$

$$\frac{dx}{da} = -5; \quad \frac{dy}{da} = 0; \quad J_c = 5$$

$$\text{Gauss point} = -0.57735; \quad \text{weight} = 1; \quad J_c = 5$$

$$N_c^T = (0 \ 0 \ 0.788675 \ 0.211325)$$

$$r_q^T = (0 \ 0 \ 0 \ 0 \ 0 \ -394.338 \ 0 \ -105.662)$$

$$\text{Gauss point} = 0.57735; \quad \text{weight} = 1; \quad J_c = 5$$

$$N_c^T = (0 \ 0 \ 0.211325 \ 0.788675)$$

$$r_q^T = (0 \ 0 \ 0 \ 0 \ 0 \ -105.662 \ 0 \ -394.338)$$

Summing contributions from all Gauss points gives us

$$r_q^T = (0 \ 0 \ 0 \ 0 \ 0 \ -500 \ 0 \ -500)$$

Complete element equations for element 1:

$$\begin{pmatrix} 54837.5 & 24509.8 & -38059 & 7731.28 & -27739.1 & -24509.8 & 10960.6 & -7731.28 \\ 24509.8 & 54837.5 & -7731.28 & 10960.6 & -24509.8 & -27739.1 & 7731.28 & -38059 \\ -38059 & -7731.28 & 54837.5 & -24509.8 & 10960.6 & 7731.28 & -27739.1 & 24509.8 \\ 7731.28 & 10960.6 & -24509.8 & 54837.5 & -7731.28 & -38059 & 24509.8 & -27739.1 \\ -27739.1 & -24509.8 & 10960.6 & -7731.28 & 54837.5 & 24509.8 & -38059 & 7731.28 \\ -24509.8 & -27739.1 & 7731.28 & -38059 & 24509.8 & 54837.5 & -7731.28 & 10960.6 \\ 10960.6 & 7731.28 & -27739.1 & 24509.8 & -38059 & -7731.28 & 54837.5 & -24509.8 \\ -7731.28 & -38059 & 24509.8 & -27739.1 & 7731.28 & 10960.6 & -24509.8 & 54837.5 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -500 \\ 0 \\ -500 \end{pmatrix}$$

Essential BC:

Note	dof	Value
1	1	0
	2	0
4	1	0
	2	0

Global matrices after incorporating EBC:

$$\mathbf{K} = \begin{pmatrix} 54837.5 & -24509.8 & 10960.6 & 7731.28 \\ -24509.8 & 54837.5 & -7731.28 & -38059 \\ 10960.6 & -7731.28 & 54837.5 & 24509.8 \\ 7731.28 & -38059 & 24509.8 & 54837.5 \end{pmatrix}$$

$$\mathbf{R}^T = (0 \ 0 \ 0 \ -500)$$

Solution of global equations:

$$\{-0.00763677, -0.020883, 0.0108145, -0.0273683\}$$

Solution for element quantities:

$$\text{nodal displacements} = (0 \ 0 \ -0.00763677 \ -0.020883 \ 0.0108145 \ -0.0273683 \ 0 \ 0)$$

Solution at $\{s, t\} = \{0, 0\} \Rightarrow \{x, y\} = \{5, 5\}$:

$$\text{interpolation functions} = \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right\}$$

$$u = 0.00079444; \quad v = -0.0120628$$

$$\boldsymbol{\epsilon}^T = (0.000158888 \ -0.000324261 \ 0.000158888 \ -0.00149 \ 0 \ 0)$$

$$\boldsymbol{\sigma}^T = (1.00188 \times 10^{-13} \ -324.261 \ 0 \ -500 \ 0 \ 0)$$

$$\text{principal stresses} = (363.499 \ 0 \ -687.76)$$

$$\text{effective stress (von Mises)} = 924.741$$

Example 5.4 The 4/1 element is used to solve the plane strain problem considered in Example 5.2. The solution at node 5 ($x = 0, y = 5$) for various Poisson's ratios is summarized in Table 5.2. For Poisson's ratios up to 0.45, both the stress and the displacement values are comparable to those in the Example 5.2. For Poisson's ratios of 0.49 and 0.499, the solution obtained by the u/p element follows the expected trend and does not show the locking behavior exhibited by the standard displacement-based element.

See Mathematica/MATLAB Implementation 5.2 on the book Web site.

