Shells

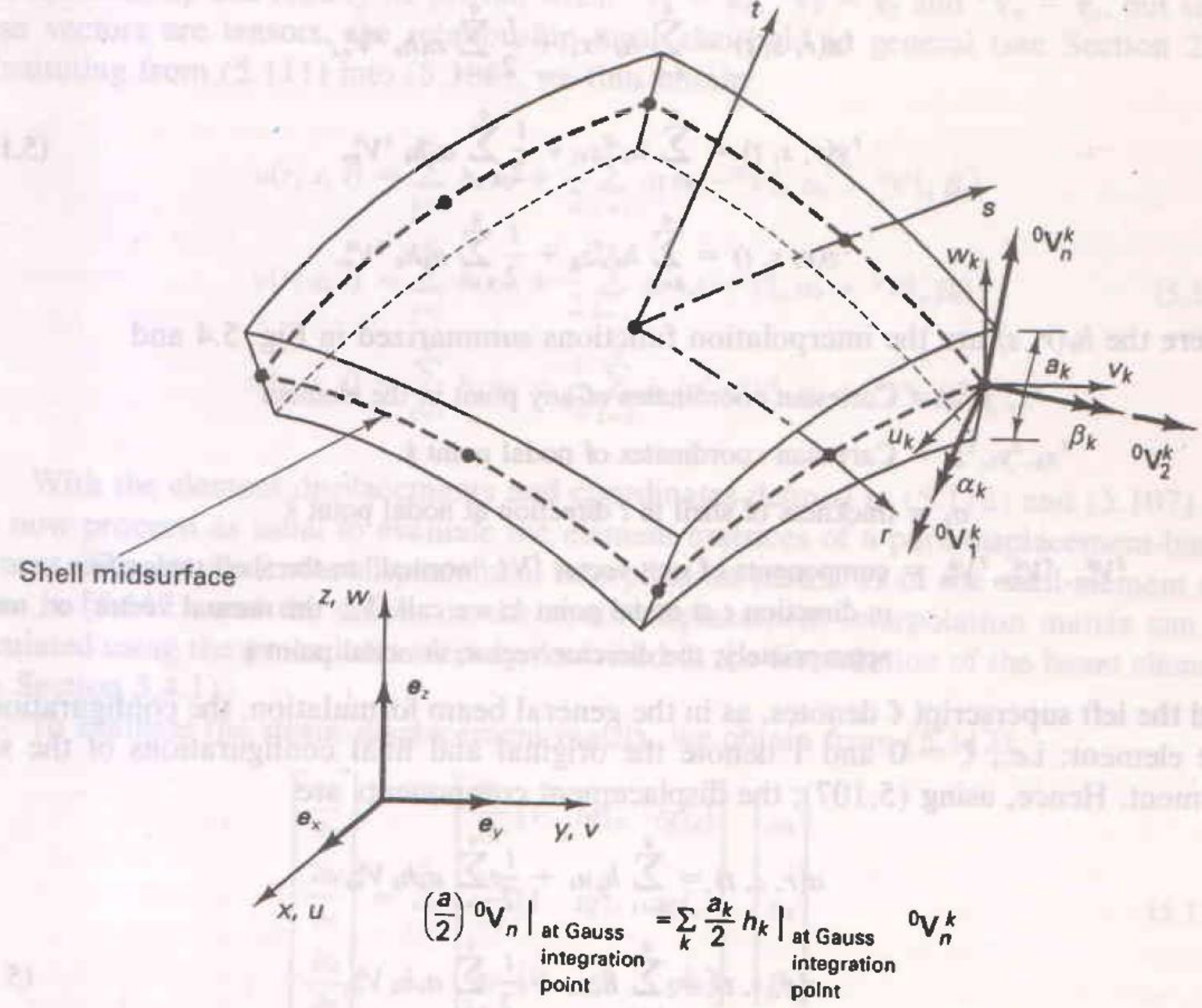


Figure 5.33 Nine-node shell element; also, definition of orthogonal $\bar{r}$, $\bar{s}$, t axes for constitutive relations

EXAMPLE 5.32: Consider the four-node shell element shown in Fig. E5.32.

(a) Develop the entries in the displacement interpolation matrix.

(b) Calculate the thickness at the midpoint of the element and give the direction in which this thickness is measured.

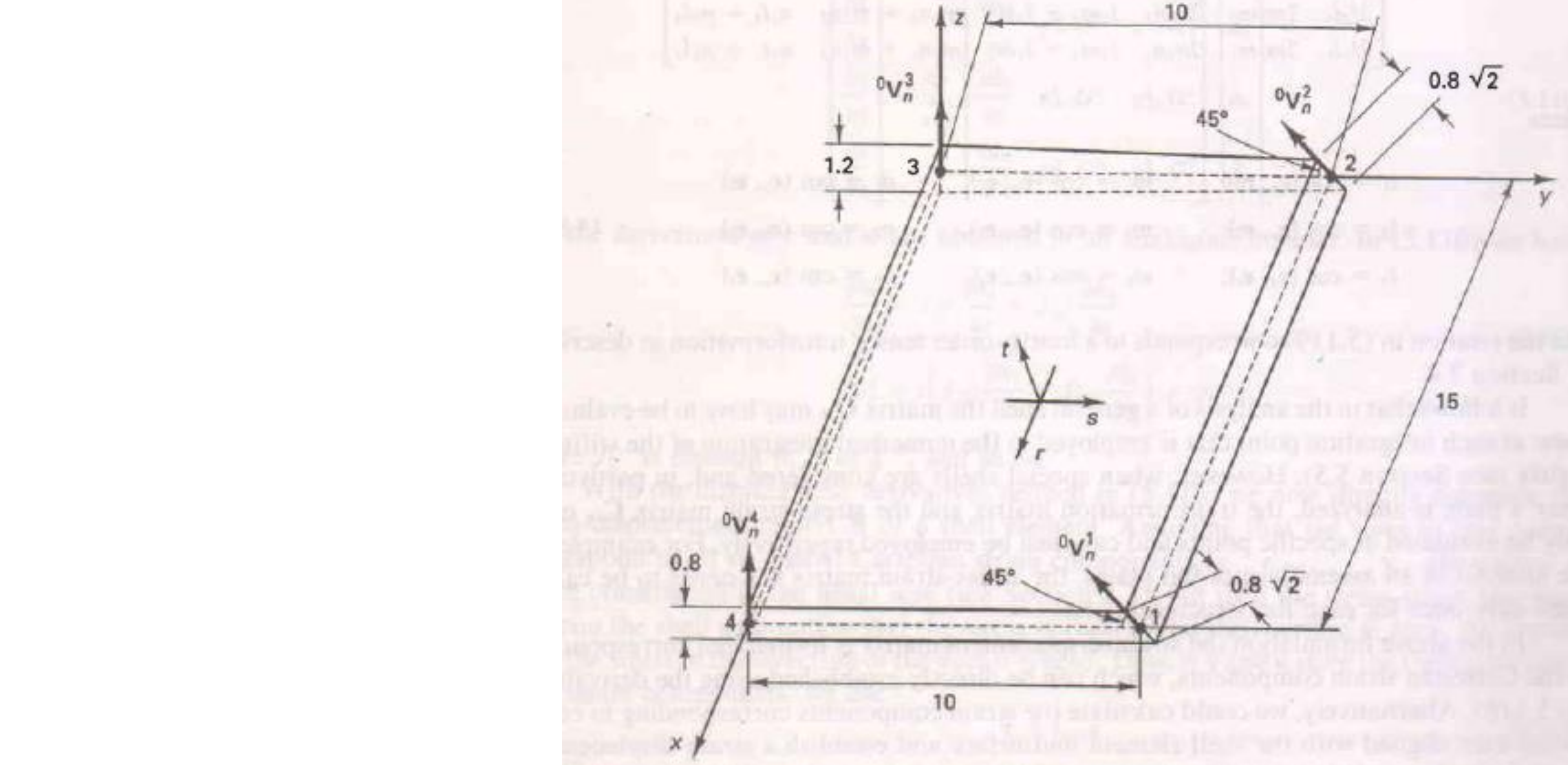


Figure E5.32 Four-node shell element

The shell element considered has varying thickness

$$ {}^0\mathbf{V}_n^1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}; \qquad {}^0\mathbf{V}_n^2 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}; \qquad {}^0\mathbf{V}_n^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \qquad {}^0\mathbf{V}_n^4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ {}^0\mathbf{V}_1^1 = {}^0\mathbf{V}_1^2 = {}^0\mathbf{V}_1^3 = {}^0\mathbf{V}_1^4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$

$$
{}^0\mathbf{V}_2^1 = {}^0\mathbf{V}_2^2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}; \qquad {}^0\mathbf{V}_2^3 = {}^0\mathbf{V}_2^4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

Also, $\quad a_1 = a_2 = 0.8\sqrt{2}; \qquad a_3 = 1.2; \qquad a_4 = 0.8$

The above expressions give all entries in (5.112).

To evaluate the thickness at the element midpoint and the direction in which the thickness is measured, we use the relation

$$
\left(\frac{a}{2}\right){}^0\mathbf{V}_n\Big|_{\text{midpoint}} = \sum_{k=1}^{4} \frac{a_k}{2} h_k\Big|_{r=s=0} {}^0\mathbf{V}_n^k
$$

where a is the thickness and the director vector ${}^0\mathbf{V}_n$ gives the direction sought. This expression gives

$$
\frac{a}{2}\,{}^0\mathbf{V}_n = \frac{0.8\sqrt{2}}{4}\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + \frac{1.2}{8}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{0.8}{8}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2 \\ 0.45 \end{bmatrix}
$$

which gives $\qquad {}^0\mathbf{V}_n = \begin{bmatrix} 0.0 \\ -0.406 \\ 0.914 \end{bmatrix}; \qquad a = 0.985$

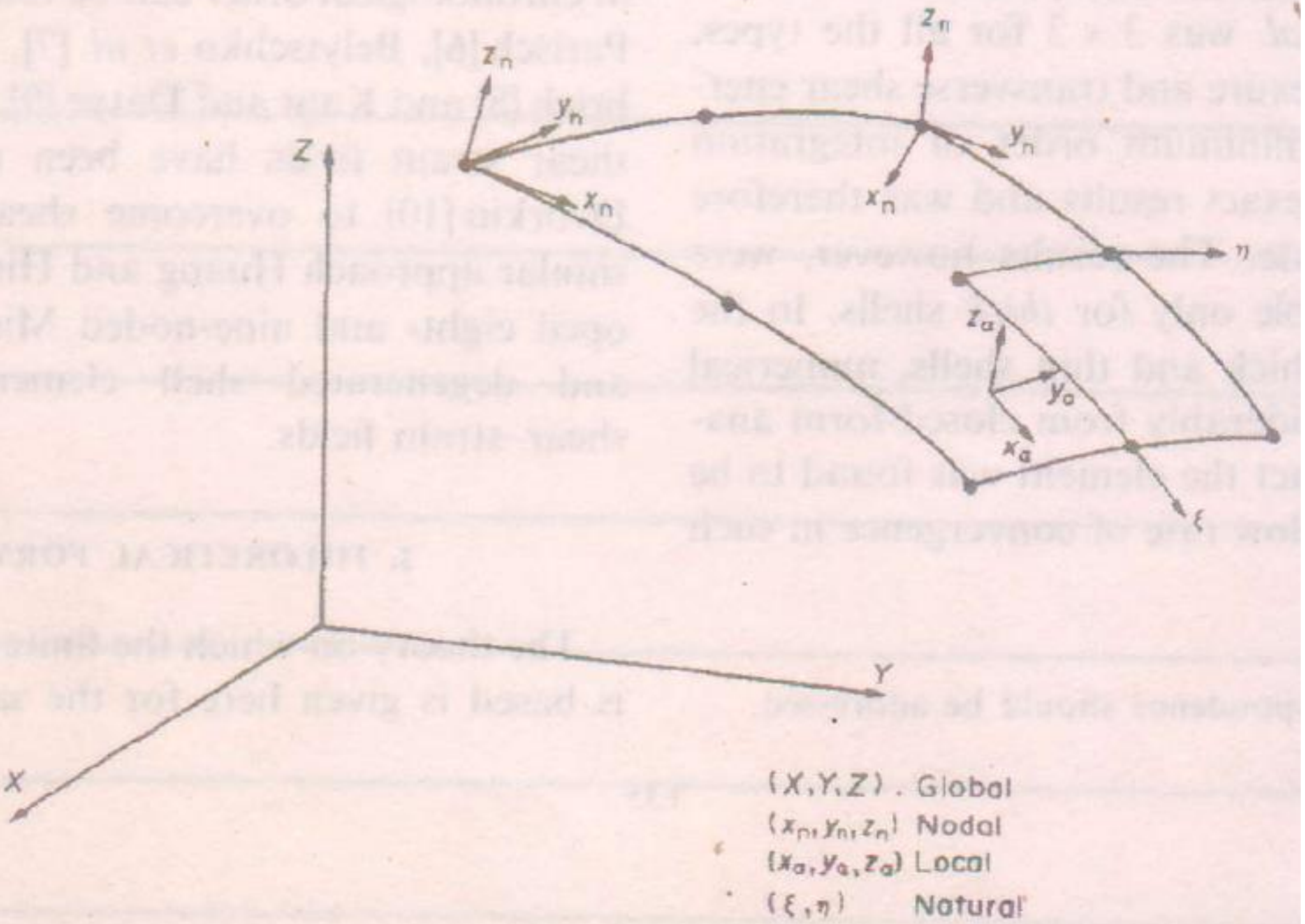


Fig. 1. Coordinate systems.

$\bar{r}, \bar{s}, t$ at any point.

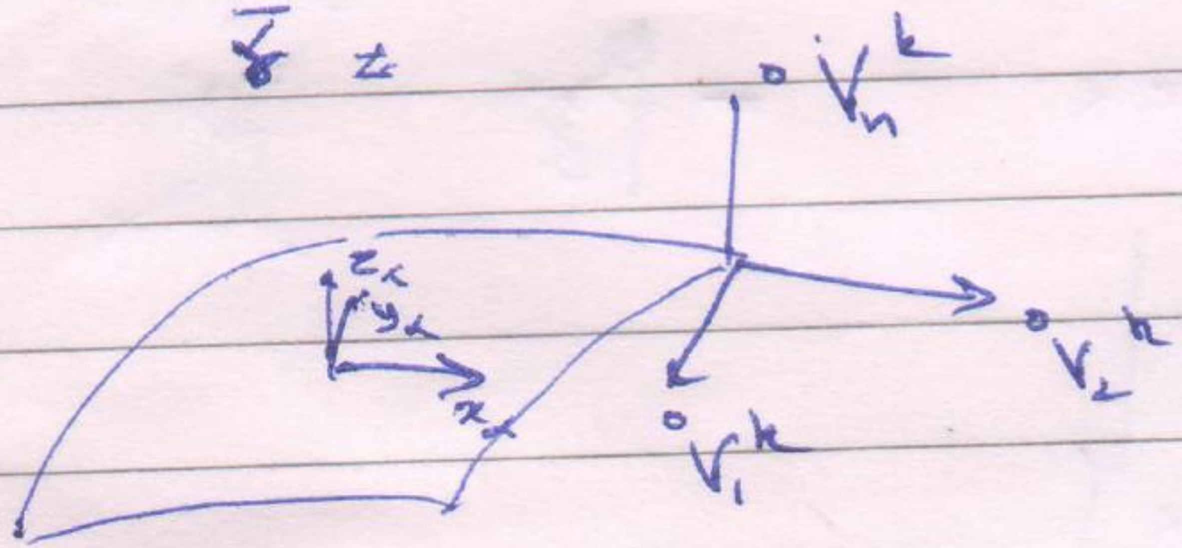
A set of tangent vectors for midsurface are

$$\vec{V}_r = \frac{\partial x}{\partial r} \underset{\rightharpoondown}{i} + \frac{\partial y}{\partial r} \underset{\rightharpoondown}{j} + \frac{\partial z}{\partial r} \underset{\rightharpoondown}{k}$$

tangent to r

$$\bar{r} \quad \underset{\rightharpoondown}{V}_{s, \rightharpoondown} = \left(\sum \frac{\partial h_k}{\partial r} x_k \right) \underset{\rightharpoondown}{i} + \left(\sum \frac{\partial h_k}{\partial r} y_k \right) \underset{\rightharpoondown}{j}$$

$$+ \left(\sum \frac{\partial h_k}{\partial r} z_k \right) \underset{\rightharpoondown}{k}$$



k_α is along $\underline{^{o}V^k}$ at that point

i_α is along $\bar{r}$

j_α is along $\underset{\rightarrow}{k_\alpha} \times \underset{\rightarrow}{i_\alpha}$

$$u_\alpha(x_\alpha, y_\alpha, z_\alpha) = u_{o\alpha}(x_\alpha, y_\alpha) - z_\alpha \beta_{x\alpha}(x_\alpha, y_\alpha)$$

$$= \sum_k h_k u_{o\alpha}^k + z_\alpha \sum h_k \theta_y^k$$

$$v_\alpha(x_\alpha, y_\alpha, z_\alpha) = v_{o\alpha}(x_\alpha, y_\alpha) - z_\alpha \beta_{y\alpha}(x_\alpha, y_\alpha)$$

$$= \sum_k h_k v_{o\alpha}^k - z_\alpha \sum h_k \theta_x^k$$

$$w_\alpha(x_\alpha, y_\alpha, z_\alpha) = w_{o\alpha}(x_\alpha, y_\alpha)$$

$$\epsilon_{x\alpha} = \epsilon_{xo\alpha} - z_\alpha \frac{\partial \beta_{x\alpha}}{\partial x_\alpha}$$

$$\epsilon_{y\alpha} = \epsilon_{yo\alpha} - z_\alpha \, \beta_{y\alpha, y\alpha}$$

$$\gamma_{xy\alpha} = \gamma_{xyo\alpha} - z_\alpha \left(\beta_{x\alpha, y\alpha} + \beta_{y\alpha, x\alpha} \right)$$

$$\gamma_{xz\alpha} = -\beta_{x\alpha} + \frac{\partial w_\alpha}{\partial x_\alpha}$$

$$\gamma_{yz\alpha} = -\beta_{y\alpha} + \frac{\partial w_\alpha}{\partial y_\alpha}$$

- At any point on shell surface there are 5 dof

$$\underset{\sim}{q_\alpha} = \begin{Bmatrix} u_{0\alpha} \\ v_{0\alpha} \\ w_{0\alpha} \\ \beta_{x\alpha} \\ \beta_{y\alpha} \end{Bmatrix} \rightarrow \text{dof}$$

Int V.W

$$= \iint \Big[\delta \epsilon_{x\alpha} \sigma_{x\alpha} + \delta \epsilon_{y} \sigma_{y\alpha} + \delta \gamma_{xy\alpha} \tau_{xy\alpha} + \delta \gamma_{xz\alpha} \tau_{xz\alpha} + \delta \gamma_{yz\alpha} \tau_{yz\alpha} \Big] \, dV$$

$$= \iint \Big\{ \sigma_{x\alpha} \left(\delta \epsilon_{x0\alpha} - z \delta \beta_{x\alpha, x\alpha} \right) + \sigma_{y\alpha} \left(\delta \epsilon_{y0\alpha} - z \delta \beta_{y\alpha, y\alpha} \right) + \tau_{xy\alpha} \Big[\delta \gamma_{xy0} - z \delta \left(\beta_{y_\alpha x_\alpha} + \beta_{x_\alpha y_\alpha} \right) \Big] + \tau_{yz\alpha} \left[\delta \gamma_{yz\alpha} \right] + \tau_{xz\alpha} \delta \gamma_{xz\alpha} \Big\} \, dz \, dA$$

$$= \int_A \delta \epsilon_{xo\alpha} N_{x\alpha} + \delta \epsilon_{oy\alpha} N_{y\alpha} + \gamma_{xyo} N_{xy\alpha}$$

$$+ M_{x\alpha} \left(-\delta \beta_{x_\alpha, x_\alpha}\right) + M_{y\alpha} \left(-\delta \beta_{y\alpha, \alpha}\right)$$

$$+ M_{xy\alpha} \left[-\left(\delta \beta_{y_\alpha, x_\alpha} + \delta \beta_{x_\alpha, y_\alpha}\right)\right] \cancel{dA}$$

$$+ \gamma_{xz} Q_{x\alpha} + \gamma_{yz} Q_{y\alpha} \Big\} dA$$

$$N_{x\alpha} = \int_{-t_\alpha/2}^{t_\alpha/2} \sigma_{x\alpha} \, dz \quad \text{etc.}$$

$$N_\alpha = \begin{Bmatrix} N_{x\alpha} \\ N_{y\alpha} \\ N_{xy\alpha} \end{Bmatrix} = \begin{bmatrix} [D_{m\alpha}]_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{Bmatrix} \epsilon_{0x\alpha} \\ \epsilon_{0y\alpha} \\ \cancel{w_{0\alpha}} \\ \gamma_{0xy,\alpha} \end{Bmatrix}$$

$$M_\alpha = \begin{Bmatrix} M_{x\alpha} \\ M_{y\alpha} \\ M_{xy\alpha} \end{Bmatrix} = \begin{bmatrix} 0_{3\times3} & [D_{b\alpha}]_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{Bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{Bmatrix}$$

$$Q_\alpha = \begin{Bmatrix} Q_{x\alpha} \\ \ldots \end{Bmatrix} = \begin{bmatrix} D_{s\alpha} & [0] & [0] \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

$$\mathbf{D}_{m2} = \begin{bmatrix} \dfrac{Et_2}{(1-vv)} & \dfrac{vEt_2}{(1-vv)} & 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{vEt_2}{(1-vv)} & \dfrac{Et_2}{(1-vv)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{Et_2}{2(1+v)} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_{b2} = \begin{bmatrix} 0 & 0 & 0 & \dfrac{Et_2^3}{12(1-vv)} & \dfrac{vEt_2^3}{12(1-vv)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{vEt_2^3}{12(1-vv)} & \dfrac{Et_2^3}{12(1-vv)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{Et_2^3}{24(1+v)} & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_{s2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{Et_2}{2.4(1+v)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{Et_2}{2.4(1+v)} \end{bmatrix}.$$

$$\bar{\epsilon}_{m\alpha} = B_{m\alpha}\, q_\alpha$$

$$\bar{\epsilon}_{b\alpha} = B_{b\alpha}\, q_\alpha$$

$$\gamma_{s\alpha} = B_{s\alpha}\, q_\alpha$$

$$q_\alpha = \begin{Bmatrix} u_{o\alpha} \\ v_{o\alpha} \\ w_{o\alpha} \\ \beta_{y\alpha} \\ \beta_{x\alpha} \end{Bmatrix}$$

$$\begin{matrix} u_{o\alpha} \\ v_{o\alpha} \\ w_{o\alpha} \\ \beta_{y\alpha} \\ \beta_{x\alpha} \end{matrix} = \sum_{i=1}^{N} \begin{bmatrix} h_i(r,s)\, u_{i\alpha} \\ h_i(r,s)\, v_{i\alpha} \\ h_i(r,s)\, w_{i\alpha} \\ -h_i(r,s)\, \theta_{x\alpha} \\ h_i(r,s)\, \theta_{y,\alpha} \end{bmatrix}$$

$$= \sum_{i=1}^{N} \begin{bmatrix} h_i & 0 & 0 & 0 & 0 \\ 0 & h_i & 0 & 0 & 0 \\ 0 & 0 & h_i & 0 & 0 \\ 0 & 0 & 0 & -h_i & 0 \\ 0 & 0 & 0 & 0 & h_i \end{bmatrix} \begin{Bmatrix} u_{i\alpha} \\ v_{i\alpha} \\ w_{i\alpha} \\ \theta_{ix\alpha} \\ \theta_{iy\alpha} \end{Bmatrix}$$

$$\underline{q}_{\alpha} = \sum_{i=1}^{N} h_i \, q_{i\alpha}$$

Let displacement at node 'i' in nodal coord system

$$q_{in} = \begin{bmatrix} u_{on} & v_{on} & w_{on} & \theta_{xn} & \theta_{yn} \end{bmatrix}$$

$$q_{i\alpha} = R_{\alpha i}\, q_{in}$$

$$q_{\alpha} = \sum_{i=1}^{N} N_i R_{\alpha i}\, q_{in}$$

$$R_{xi} = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_n & \mathbf{i}_x \cdot \mathbf{j}_n & \mathbf{i}_x \cdot \mathbf{k}_n & 0 & 0 \\ \mathbf{j}_x \cdot \mathbf{i}_n & \mathbf{j}_x \cdot \mathbf{j}_n & \mathbf{j}_x \cdot \mathbf{k}_n & 0 & 0 \\ \mathbf{k}_x \cdot \mathbf{i}_n & \mathbf{k}_x \cdot \mathbf{j}_n & \mathbf{k}_x \cdot \mathbf{k}_n & 0 & 0 \\ 0 & 0 & 0 & \mathbf{i}_x \cdot \mathbf{i}_n & \mathbf{i}_x \cdot \mathbf{j}_n \\ 0 & 0 & 0 & \mathbf{j}_x \cdot \mathbf{i}_n & \mathbf{j}_x \cdot \mathbf{j}_n \end{bmatrix}$$

$$\epsilon_{m\alpha} = L_{m\alpha}\, q_{\alpha} = L_{m\alpha} \sum_{i=1} h_i\, q_{i\alpha}$$

$$= \sum_{i=1}^{N} B_{mi\alpha}\, q_{i\alpha}$$

$$= \sum_{i=1}^{N} B_{mi\alpha}\, R_{i\alpha}\, q_{in}$$

$$= \sum_{i=1}^{n} B^{*}_{mi}\, q_{in} = [B^{*}_{m}]\{q_{n}\}$$

Similarly,

$$\epsilon_{b\alpha} = [B_b^*] \{q_n\}$$

$$\epsilon_{s\alpha} = [B_s^*] \{q_n\}$$

$$\bar{\epsilon}_{\alpha} = [B^*]\{q_n\}$$

$$= [B_m^* \quad B_b^* \quad B_s^*]^T \{q_n\}$$

$$\mathbf{L}_{m2} = \begin{bmatrix} \dfrac{\partial}{\partial x_2} & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y_2} & 0 & 0 & 0 \\ \dfrac{\partial}{\partial y_2} & \dfrac{\partial}{\partial x_2} & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{L}_{b2} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dfrac{\partial}{\partial x_2} \\ 0 & 0 & 0 & -\dfrac{\partial}{\partial y_2} & 0 \\ 0 & 0 & 0 & -\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial y_2} \end{bmatrix}$$

$$\mathbf{L}_{s2} = \begin{bmatrix} 0 & 0 & \dfrac{\partial}{\partial x_2} & 0 & 1 \\ 0 & 0 & \dfrac{\partial}{\partial y_2} & -1 & 0 \end{bmatrix}$$

You may notice that we have ignored a sixth row in R matrix. Actually, it should also be there and would transform the rotation $\theta_{x\alpha}$ and $\theta_{y\alpha}$ along the normal at the node. The normal at a node does not really coincide with normal at $(x_\alpha, y_\alpha, z_\alpha)$. However this component of rotation along node normal may be small as angle between the node normal and normal at $(x_\alpha, y_\alpha, z_\alpha)$ may be small. By rotating to nodal coordinate system continuity of variables $(u_n, v_n, w_n, \alpha, \beta)$ is maintained.

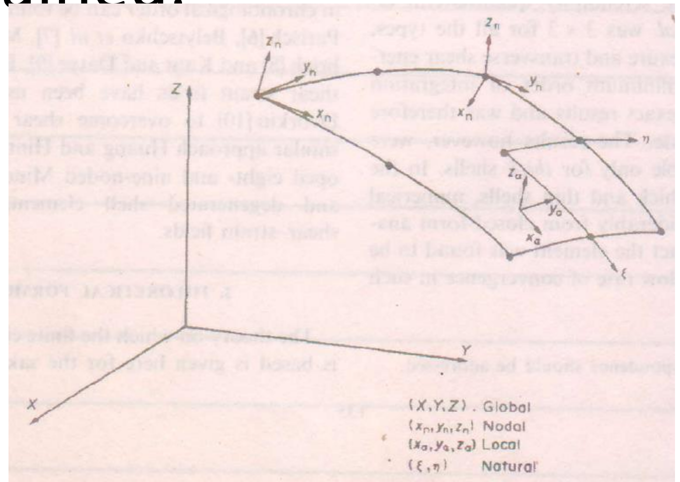


Fig. 1. Coordinate systems.

- Instead of rotating to nodal coordinate system we can also rotate to global coordinate system and we have six dofs.

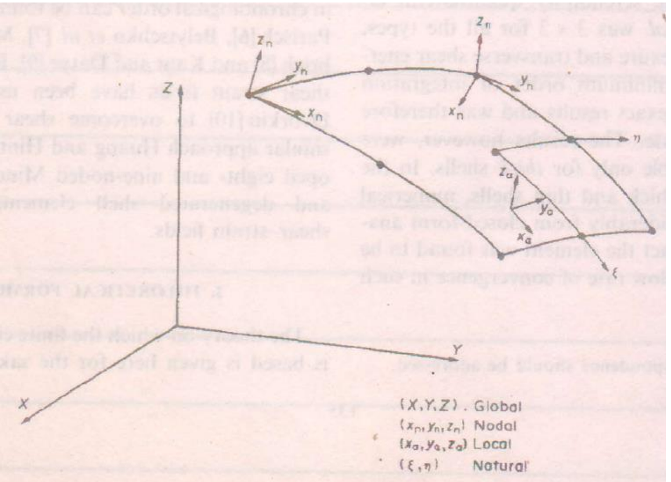


Fig. 1. Coordinate systems.

$$\mathbf{K}^e = \int_A \mathbf{B}^{*t} \bar{\mathbf{D}}_z^e \mathbf{B}^* \, \mathrm{d}A$$

$$\mathbf{K}^e = \mathbf{K}_m^e + \mathbf{K}_b^e + \mathbf{K}_s^e,$$

$$\mathbf{K}_m^e = \int_{-1}^{+1} \int_{-1}^{+1} \bar{\mathbf{B}}_m^{*t} \bar{\mathbf{D}}_{mz} \bar{\mathbf{B}}_m^* |\mathbf{J}| \, \mathrm{d}\xi \, \mathrm{d}\eta$$

$$\mathbf{K}_m^e = \sum_{\beta=1}^{m} w_\beta (\bar{\mathbf{B}}_m^*)_\beta^t (\bar{\mathbf{D}}_{mz})_\beta (\bar{\mathbf{B}}_m^*)_\beta |\mathbf{J}|_\beta,$$

$$\mathbf{K}_b^e = \sum_{\beta=1}^{m} w_\beta (\bar{\mathbf{B}}_b^{*t})_\beta (\bar{\mathbf{D}}_{bz})_\beta (\bar{\mathbf{B}}_b^*)_\beta |\mathbf{J}|_\beta$$

$$\mathbf{K}_s^e = \sum_{\beta=1}^{m} w_\beta (\bar{\mathbf{B}}_s^{*t})_\beta (\bar{\mathbf{D}}_{sz})_\beta (\bar{\mathbf{B}}_s^*)_\beta |\mathbf{J}|_\beta.$$

Here $\mathbf{K}_m^e$, $\mathbf{K}_b^e$, $\mathbf{K}_s^e$ are the element membrane, bending and shear stiffness matrices, respectively, m is the number of Gauss points and w_β are the weights of Gauss points. Thus, any order of integration (either 2×2 or 3×3) can be independently applied to calculate any of the individual K matrix.

Load vector

Body forces

$$\int [u \quad v \quad w] \begin{bmatrix} -\rho g_{x\alpha} \\ -\rho g_{y\alpha} \\ -\rho g_{z\alpha} \end{bmatrix} dx_\alpha \, dy_\alpha \, dz_\alpha$$

$$= \int \begin{bmatrix} u_{0\alpha} & v_{0\alpha} & w_{0\alpha} \end{bmatrix} \begin{bmatrix} -\rho g_{x\alpha} \\ -\rho g_{y\alpha} \\ -\rho g_{z\alpha} \end{bmatrix} dx_\alpha \, dy_\alpha \, dz_\alpha$$

$$+ \underbrace{\iint_{-t/2}^{t/2} \cancel{\begin{bmatrix} -z\beta_x & -z\beta_y & 0 \end{bmatrix}} \begin{bmatrix} \\ \\ \end{bmatrix}}_{0} dz_\alpha$$

$$\left(\because \frac{z^2}{2} \Big|_{-t/2}^{t/2} = 0 \right)$$

$$= \int \begin{bmatrix} u_{0\alpha} & v_{0\alpha} & w_{0\alpha} \end{bmatrix} \begin{bmatrix} -\rho g_{x\alpha} \\ -\rho g_{y\alpha} \\ -\rho g_{z\alpha} \end{bmatrix} t \, dx_\alpha \, dy_\alpha$$

$$= \int \begin{bmatrix} & q_\alpha & \end{bmatrix} \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_1 & 0 \\ 0 & 0 & h_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ h_2 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_2 \\ \vdots & \vdots & \end{bmatrix} \begin{Bmatrix} -\rho g_{x\alpha} \\ -\rho g_{y\alpha} \\ -\rho g_{z\alpha} \end{Bmatrix} t \, dx_\alpha \, dy_\alpha$$

Similarly for pressure

$$\int [q_\alpha] \begin{Bmatrix} 0 \\ 0 \\ h_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ h_2 \\ \vdots \end{Bmatrix} p \, dx_\alpha \, dy_\alpha$$

The shape fns are in terms of (r, s) coord. whereas integration is in terms of x_α, y_α, so we need Jacobian. Also $i_\alpha, j_\alpha, k_\alpha$ ~~etc~~ are related to i, j, k.

$$J = \begin{vmatrix} \frac{\partial x_\alpha}{\partial r} & \frac{\partial y_\alpha}{\partial r} \\ \frac{\partial x_\alpha}{\partial s} & \frac{\partial y_\alpha}{\partial s} \end{vmatrix}$$

$$|J| = \begin{vmatrix} l_{1\alpha}\dfrac{\partial x}{\partial r} + m_{1\alpha}\dfrac{\partial y}{\partial r} + n_{1\alpha}\dfrac{\partial z}{\partial r} & l_{2\alpha}\dfrac{\partial x}{\partial r} + m_{2\alpha}\dfrac{\partial y}{\partial r} + n_{2\alpha}\dfrac{\partial z}{\partial r} \\ l_{1\alpha}\dfrac{\partial x}{\partial s} + m_{1\alpha}\dfrac{\partial y}{\partial s} + n_{1\alpha}\dfrac{\partial z}{\partial s} & l_{2\alpha}\dfrac{\partial x}{\partial s} + m_{2\alpha}\dfrac{\partial y}{\partial s} + n_{2\alpha}\dfrac{\partial z}{\partial s} \end{vmatrix}$$

- The coordinates x, y in terms of nodal coordinates are given below. On the mid-plane put t=0 in the expressions below.

- Loads at the boundary due to moments or transverse loads can be resolved into moments or forces in x, y, z directions.

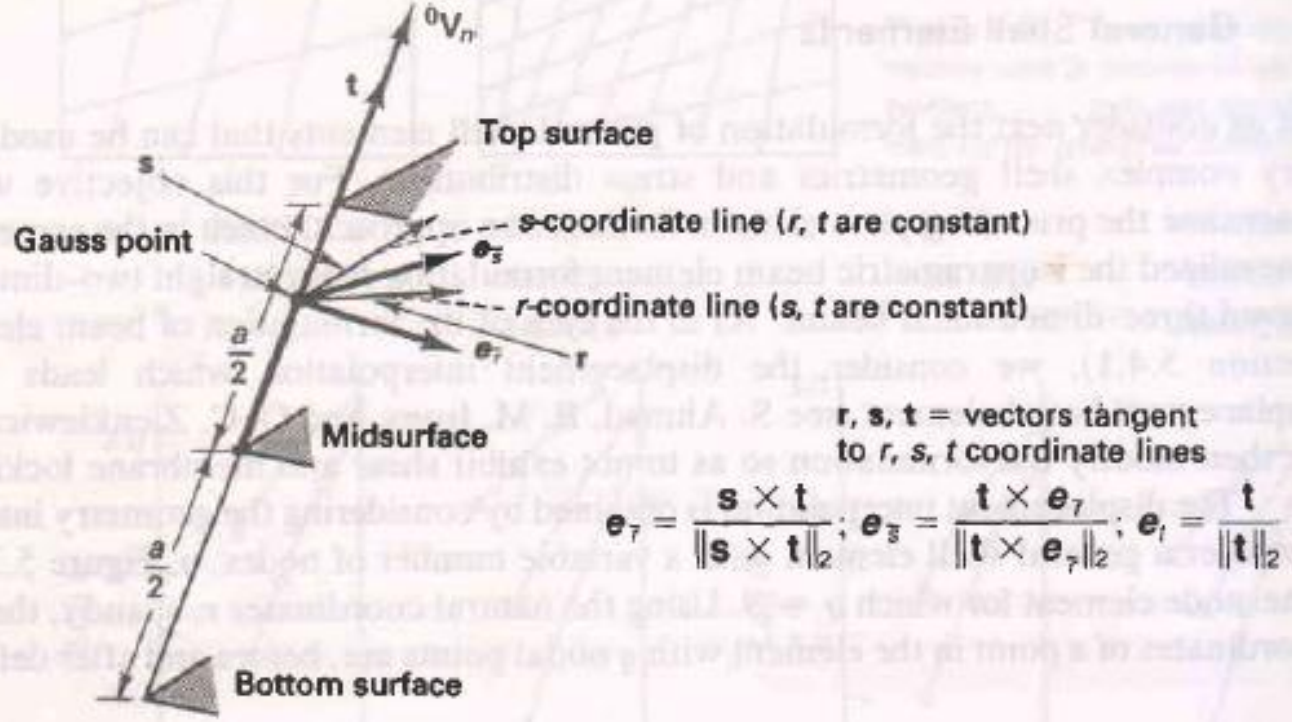


Figure 5.33 (*continued*)

$$
\begin{aligned}
{}^{\ell}x(r, s, t) &= \sum_{k=1}^{q} h_k \, {}^{\ell}x_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{nx}^{k} \\
{}^{\ell}y(r, s, t) &= \sum_{k=1}^{q} h_k \, {}^{\ell}y_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{ny}^{k} \qquad (5.107) \\
{}^{\ell}z(r, s, t) &= \sum_{k=1}^{q} h_k \, {}^{\ell}z_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k \, {}^{\ell}V_{nz}^{k}
\end{aligned}
$$

$^{\ell}x, {}^{\ell}y, {}^{\ell}z$ = Cartesian coordinates of any point in the element

$^{\ell}x_k, {}^{\ell}y_k, {}^{\ell}z_k$ = Cartesian coordinates of nodal point k

a_k = thickness of shell in t direction at nodal point k

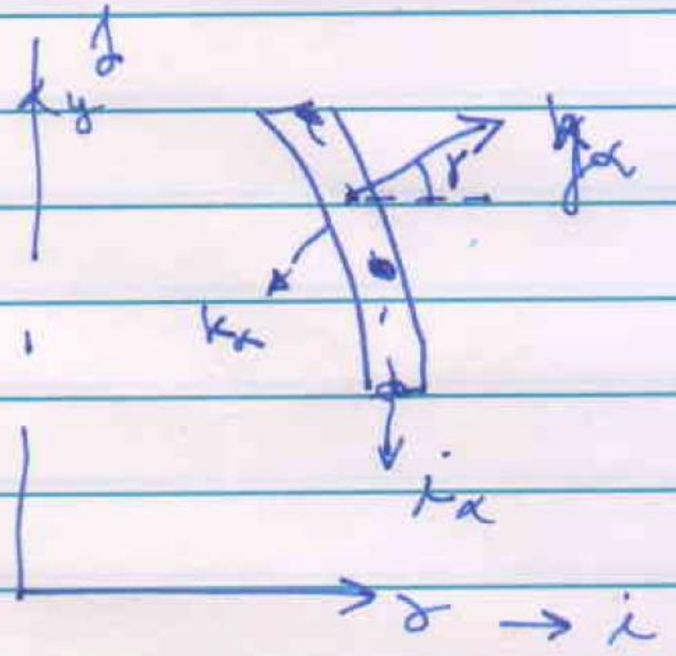
$^{\ell}V_{nx}^k, {}^{\ell}V_{ny}^k, {}^{\ell}V_{nz}^k$ = components of unit vector $^{\ell}\mathbf{V}_n^k$ "normal" to the shell midsurface in direction t at nodal point k; we call $^{\ell}\mathbf{V}_n^k$ the normal vector[7] or, more appropriately, the director vector, at nodal point k

$\ell = 0$ and 1 denote the original and final configurations of the shell

$$
\begin{aligned}
u(r, s, t) &= \sum_{k=1}^{q} h_k u_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{nx}^k \\
v(r, s, t) &= \sum_{k=1}^{q} h_k v_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{ny}^k \\
w(r, s, t) &= \sum_{k=1}^{q} h_k w_k + \frac{t}{2} \sum_{k=1}^{q} a_k h_k V_{nz}^k
\end{aligned}
\tag{5.108}
$$

Axisymmetric shell (HW)

For a 3-node axisymmetric shell element



$$\underset{\sim}{\gamma}_{\alpha} \cdot \underset{\sim}{i} = \cos \gamma$$

etc.

Let θ_α be the rotation of the normal about the k_α axis

k_α is perpendicular to plane of paper

$$u_\alpha = u_{o\alpha} - z_\alpha \beta_{x\alpha}$$

$$w_\alpha = W_{o\alpha}(x_\alpha)$$

$$\{q_\alpha\} = \begin{Bmatrix} u_{o\alpha} \\ W_{o\alpha} \\ \beta_{x\alpha} \end{Bmatrix}$$

Following the procedure followed for shell element, determine the membrane and bending strain in r-y coordinate system. Hence determine the stiffness matrix.

Beam K.E and Mass Matrix

$$= \frac{1}{2} \int \rho (\dot{u}^2 + \dot{w}^2) \, dV$$

$$= \frac{1}{2} \int \rho (\dot{u}^T \dot{u} + \dot{w}^T w) \, b \, dz \, dx$$

$$= \frac{1}{2} \int_L \int_{-h/2}^{h/2} \rho \left(-z \frac{d\dot{\beta}}{dx}\right)^T \left(-z \frac{d\dot{\beta}}{dx}\right) b \, dz \, dx$$

$$+ \frac{1}{2} \int_L \int_{-h/2}^{h/2} \rho \, \dot{w}^T \dot{w} \, b \, dz \, dx$$

$$= \frac{1}{2}\iint \rho z^2 \begin{bmatrix}\dot{w}_1 & \dot{w}_2 & \dot{\beta}_1 & \dot{\beta}_2\end{bmatrix} H_B^T H_B \begin{bmatrix}\dot{w}_1 \\ \dot{w}_2 \\ \dot{\beta}_1 \\ \dot{\beta}_2\end{bmatrix} b\, dz\, dx$$

$$+ \frac{1}{2}\int \rho \begin{bmatrix} & \end{bmatrix} H_\omega^T H_\omega \begin{bmatrix} \\ \end{bmatrix} b\, dz\, dx$$

$$M = \frac{1}{2}\int \frac{b\rho h^3}{12} \begin{bmatrix} & \end{bmatrix} H_B^T H_B \begin{bmatrix} \\ \end{bmatrix} dx + \frac{1}{2}\int \rho h \begin{bmatrix} & \end{bmatrix} H_w^T H_w \begin{bmatrix} \\ \end{bmatrix} dx$$

$$M = \int \begin{bmatrix} H_w \\ H_B \end{bmatrix}^T \begin{bmatrix} \rho b h & \\ & \frac{\rho b h^3}{12} \end{bmatrix} \begin{bmatrix} H_w \\ H_b \end{bmatrix} |\det J|\, dr$$

HW

Using similar approach for shell show that

$$E = \frac{1}{2} \int \dot{\underset{\sim}{u}}^T \rho \, \dot{\underset{\sim}{u}} \, dV$$

$$= \frac{1}{2} \int \dot{q}_\alpha^T \, M \, \dot{q}_\alpha \, dA$$

$$\mathbf{m} = \begin{bmatrix} I_1 & 0 & 0 & 0 & I_2 \\ 0 & I_1 & 0 & -I_2 & 0 \\ 0 & 0 & I_1 & 0 & 0 \\ 0 & -I_2 & 0 & I_3 & 0 \\ I_2 & 0 & 0 & 0 & I_3 \end{bmatrix}$$

$I_1 = \int \rho \, dz$; $I_2 = \int \rho z \, dz$; $I_3 = \int \rho z * z \, dz$.